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CONTROLLABILITY, INVARIANCE AND INTERACTION

IN LINEAR DYNAMICAL SYSTEMS

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SUMMARY

The objective of the research described in this dissertation was to investigate the relation between controllability and invariance of states and outputs of a system, and to apply the criteria of controllability and invariance to the design of systems of given interaction specifications. In order not to obscure by excessive detail the main course of this presentation, the investigation was essentially limited to linear time-invariant dynamical systems, but the extension to time variant systems is, at least conceptually, a straightforward procedure. The feasibility of such an extension has been illustrated by an example. Furthermore, only continuous time systems were considered, but the transition to discrete time systems is even less complicated.

Roughly speaking, the term "controllability" means that it is possible to transfer any initial state of the system to any final state, or change any initial output of the system to any desired final output in some finite time (properties to be referred to as state and output controllability, respectively). "Observability" means that the initial that the initial state of the system can be found from a suitable measurement of the output. State or output invariance denotes essentially the property of the system that certain states or outputs are in no way affected by some or all of the inputs.

The relation between state controllability and observability has been established by Kalman in his well-known duality theorem, but little was done to clarify the relation between controllability and invariance as well as between observability and invariance. It is shown in this

work that such relationships exist, and the derivation of these relationships is presented as part of the overall research results.

The proposed approach of analysis and synthesis of interacting systems is based on the concepts of selective controllability, selective observability, and selective invariance. These concepts are defined within the framework of more general types of controllability, observability, and invariance. Since certain types of controllability, observability, and invariance imply and/or are implied by the other, structures of these implicational relationships were developed and described. Necessary and sufficient criteria for elective controllability, observability, and invariance were derived. For the rest, such criteria were derived only if they were not readily available in subject literature.

The major result of the investigation is the development of a methodology for (1) analysis of interacting dynamical systems and (2) design of dynamical systems with specified interaction or non-interaction characteristics. A set of variables (input, output, or state) of a dynamical system is defined as "non-interacting" if and only if each of the variables in the set is selectively invariant with respect to all other variables in the set. Otherwise it is interacting. A variable is defined as strictly interacting with another variable or a set of other variables if it is controllable by the latter. The analysis and design method which has been developed is based on the systematic application of the criteria of controllability and invariance to subsets of input, state, or output variables, and the flexibility and power of the proposed approach is demonstrated by a number of examples.

CHAPTER I

INTRODUCTION AND PRECEDING DEVELOPMENTS

1.1 Controllability and Observability. The concepts of controllability and observability play a central role in modern control theory. They appear as necessary and sometimes sufficient conditions for the existence of a solution to most control problems.

Let us consider an object with states $\underline{x}(t)$, where t is time and $\underline{x}$ is an n-vector or $\underline{x}$ is an element of functional space. The object is being acted on by some control or disturbance $\underline{u}(t)$, which can be either force applied to a dynamic system, or a signal which also determines the flow of information about the object, or some other input. The state of the object $\underline{x}(t)$ is related to the input $\underline{u}(t)$ by the law of motion. For example, the law of motion might have the form of an ordinary differential equation $\dot{\underline{x}}(t) = f(\underline{x}, \underline{u}, t)$. Further it is stipulated that information about the behavior of the object acted on by $\underline{u}(t)$ is available in terms of the (vector) function $y(t)$, which is referred to as the output of the system.

Controllability deals essentially with the problem of the existence of some allowable control $\underline{u}\ [t_o, t_f]$ which transfers the system from the state $\underline{x}(t)_o = \underline{x}_o$ to the state $\underline{x}(t_f) = \underline{x}_f$, or changes the output of the system from $\underline{y}(t_o) = \underline{y}_o$ to $\underline{y}(t_f) = \underline{y}_f$, whereas observability deals with the identification of states from output measurements. In other words, a system which is not controllable has dynamic modes of behavior which depend solely on initial conditions or disturbance inputs. A system which is not observable has dynamic modes of behavior which cannot be ascertained from measurement of the available outputs [98].

The problem is not only of theoretical interest, but also of great practical importance. In regulator and tracking problems, it is equivalent to the problem of selecting initial conditions which would make it possible to reach certain desired point $\underline{x}_f$ in the state space or $\underline{y}_f$ in the output space. For great many optimal control problems the solution, if known at all, is very difficult to implement. Thus, in the theory of optimal processes and in its applications, the existence of a control $\underline{u}\,[t_o, t_f]$ which transfers the system from the state $\underline{x}_o$ to the state $\underline{x}_f$ is a very important consideration even if one does not take into account the requirements of optimization according to some criteria.

Certain postulates bearing on the idea of controllability and the closely related theory of observability appeared in the work of Pontryagin [220] about 1956 as a technical requirement in the study of the minimal-time control systems. The concept of controllability is used also in a similar context by La Salle [162]. In a more general form this condition is contained in a paper of Chow [54] who studied the differential-geometric analog of controllability. Further leads in this direction appear in early works of Bellman [25], Gamkrelidze [93], Kulikowski [155] and other authors on the problem of limiting speed of response in optimal control.

The related minimal energy problem has been considered by Kulikowski [154], Lee [165] and many others. The first solution of this problem (in the discrete case) in terms of the so-called W matrix was obtained in 1958 by Bertram and Sarachik [28]. The same tool was used also by Ho [106].

In their present formulation, the concepts of controllability and observability were introduced by Kalman [114, 116] in 1959. An early hint of this concept appears in connection with the discussion of "dead-beat" sampled data systems in 1957 [113], but at that time Kalman did not yet realize that controllability plays a fundamental role in the study of control problems of all types.

The important theorem relating Jordan canonical form to controllability was noted by many people, but a rigorous proof of that theorem was first given by Kalman, Ho and Narendra [118] in 1961. A less abstract though longer proof was given by Ho [105], also Nelson [197].

The theory of controllability and observability was developed by Kalman directly on the basis of finite-dimensional linear algebra. It can be approached also via functional analysis by interpreting the problem of control and observation for linear systems as a problem of constructing linear operators, a procedure well known and extensively studied in this branch of mathematics. The problem is identical with what is known in functional analysis as the problem of moments [5, 6, 73]. This approach was developed by Krasovskii [136, 140], Fattorini [82, 83] and others.

A different approach to the problem of controllability, which is of geometrical, or, rather, set-theoretical nature, was taken, by Antosiewicz [11, 12], Conti [61, 62] and some other authors. This approach has the advantage of being applicable also to problems, such as approximate controllability, whose nature is essentially nonalgebraic. The concept of approximate controllability is somewhat more general than the notion of strict controllability, which it includes as a special case. The necessary and sufficient conditions for approximate controllability were derived by Antosiewicz [12], who also showed that if a system is approximately controllable, then there is a least time interval over which approximate control can be achieved.

Of great practical interest are controllability problems in which control inputs are constrained. Necessary and sufficient controllability conditions for such systems were derived by Antosiewicz [11], also Smith [261].

The conditions for controllability of continuous-time systems are, in essence, analogous to the conditions for controllability of discrete-time systems. In fact, most of the results for the discrete-time case can be obtained from the continuous-time by replacing the continuous time variable t with discrete points kT on the t-axis, and by replacing integrals with summations. There are few exceptions to this rule, and these were noted by Sarachik and Kreindler [243].

However, there is one important problem which arises in discrete-time systems but not in continuous-time systems. As it was pointed out by Kalman, Ho, and Narendra [118], introduction of sampling is a restriction on the possible controls $\underline{u}(t)$, so that sampling cannot improve controllability. As to the possibility that sampling might destroy controllability, it was known that this depends on the sampling frequency $1/T$, where T is some constant sampling period, and that the loss of controllability can always be avoided by choosing the sampling frequencies 1/T sufficiently large. Physically, this means that the periodicity inherent in sampling is not allowed to interact with the natural frequencies of the system to be controlled [116].

It is revealing to show at this point the relation of controllability problems to what is today usually called a "dead-beat" system. In a discussion of a now well-known paper by Bergen and Ragazzini [27], it was pointed out in 1954 that it is possible to design a sampled-data controller for any single input-single output linear stationary plant in such a way that the error in response to a step input is identically zero after a finite length of time. Kalman has shown in this context that a system is completely controllable if and only if there exists a sampled-data controller which has "dead-beat" response [116].

Sarachik and Kreindler [243] considered linear discrete-time systems which are more general than those investigated by Kalman, and refined the methods of defining controllability for systems with direct transmission from input to output, systems with singular transition matrix (i.e. those not necessarily obtained from sampling differential equations), and systems with delays.

Controllability as defined in terms of the state is neither necessary nor sufficient condition in certain types of problems of controlling the outputs of the system. This led to the definition of the concept of controllability in terms of the outputs [242, 28, 142]. Essentially, output controllability implies that any final output of the system can be attained starting with some initial conditions. It is closely related to what is called point-wise reproducibility by Brockett and Mesarovic [38]. In their terminology, reproducibility refers to the ability of a system to achieve, with its outputs, something which is desired of it. For example, functional reproducibility refers to the capabilities of a system with respect to the generation of specified time functions; asymptotic reproducibility refers to the possibility of approaching a desired behavior with increasing time; point-wise reproducibility refers to the possibility of achieving a desired value of the outputs at some point in time. Although certain of the requirements which are placed on the outputs to achieve reproducibility are similar to those frequently imposed on the state to ensure controllability, others are different as, for example, in the case of functional reproducibility. Actually the approach taken by Brockett and Mesarovic is an outgrowth of the general studies on interaction in multivariable systems [187, 188]. Systems which are not reproducible correspond to those which have been referred to as having unit-interaction [189].

A natural extension of controllability theory of linear systems to that of nonlinear systems is via linear approximations of nonlinear systems. The relation between controllability of nonlinear systems and controllability of linear approximations of these systems was investigated by several authors including Kalman [119], and Al'brekht and Krasovskii [7].

Further contributions to the controllability theory of nonlinear systems were made by Krasovskii [137], Lee and Markus [166], Rakovshchik [224], Rosen [229] and others. Hermes [103] investigated the relation between controllability and singular problems, with control appearing linearly. However, for the general case of nonlinear systems, this problem has not been carefully explored.

Serious difficulties occur if equations of motion are stochastic rather than deterministic. Fundamental questions of stochastic controllability were taken up only recently by Aoki [13], Kushner [154], Conners [60], and others, although first attempts to attack this problem are due to Kalman [120] as well. In essence, the motivation is analogous with the deterministic case: if an initial state of a given random dynamical process can be transferred (in a stochastic sense) to any desired state in a finite length of time by some control action, then this initial state is stochastically controllable. The essential difference in the treatment of the problem is that, in the stochastic case, the question of controllability cannot be resolved solely by an analysis of the algebraic aspects of the problem, but that predominant consideration is to be given to the probabilistic aspects.

Most results regarding observability of a system are directly obtainable from state controllability due to the duality relation (in an abstract algebraic sense) formulated by Kalman [114]. In essence, the duality principle says that a system is completely observable if and only if the adjoint plant is completely state controllable. Its interpretation in terms of functional analysis was first given by Krasovskii [140]. He showed that functional analytic statement of controllability is a more powerful one because many facts of the theory of optimal processes follow directly from it. Thus the functional analytic formulation actually contains the theory of limiting speed of response in linear systems, the theorem about the existence of admissible controls and the theorem about the existence of a minimal control in the set of admissible controls.

There is no meaningful dual of the concept of output controllability, but Kreindler and Sarachik [142] proposed output-predictability as a meaningful counterpart. An unforced system is called "output predictable" on some time interval $[t_o, t_f]$ if the output $\underline{y}(t)$ due to some initial state $\underline{x}(t_o)$ can be determined for all $t > t_f$ from the knowledge of $\underline{y}(t)$ on $[t_o, t_f]$. Complete observability on the interval $[t_o, t_f]$ is sufficient for output-predictability on $[t_o, t_f]$, but it is not necessary, and since many control problems can be formulated in terms of the output $\underline{y}(t)$ rather than state $\underline{x}(t)$, output controllability is less restrictive requirement in solving such problems.

There is a close relation between the concepts of controllability and observability on one hand, and the realization of a system on the other. A system which can be identified as having generated a given (experimentally observed) impulse response matrix, is said to be the realization of that impulse response. It is an irreducible realization if the dimension of its state space is minimal. The main facts relating realization to controllability are:

(i) The impulse-response matrix of a linear dynamical system depends solely on completely controllable and completely observable part of the system;

(ii) Any two completely controllable and completely observable realizations of a linear dynamical system are algebraically equivalent.

(iii) A realization of a linear dynamical system is irreducible if and only if at all times it consists of completely controllable and completely observable part of the system alone; thus every irreducible realization of a system is completely controllable and completely observable [122].

Hence, it follows that the knowledge of the impulse-response matrix identifies the completely controllable and completely observable part, and this part alone, of the dynamical system which generated it. This part is itself a dynamical system and has the smallest dimension among all realizations of the system. Moreover, this part is identified uniquely up to algebraic equivalence.

These results greatly affect the traditional approach to the analysis and synthesis of multivariable systems based on transfer function matrix rather than the differential equations. From the above statements on realization based on investigations by Kalman, Gilbert and others one can conclude that a transfer function matrix represents the controllable and observable part of the system only. Furthermore, it can be shown that controllability and observability of subsystems does not assure the controllability and observability of a composite system. Thus, transfer function matrices may satisfactorily represent all the dynamic modes of the subsystems but fail to represent all those of the composite system. The loss of hidden response modes is not easy to detect because of the complexity of the transfer function matrices and matrix algebra. As Gilbert stated, "differential equations arise naturally in relating the physical properties of a system to its response characteristics, and any mathematical procedure which neglects information contained in these equations should be viewed sceptically" [98].

Kalman also indicated that there is a parallelism between the definition of realization and the theorem of E.F. Moore on indistinguishable machines in the theory of finite automata [122]. This theorem says that the class of all machines which are indistinguishable from a given strongly connected machine S by any single experiment has a unique (up to isomorphism) member with a minimal number of states. This unique machine, called the reduced form of S, is strongly connected and has the property that any two of its states are distinguishable.

It is easy to see that "indistinguishable machines" in Moore's terminology correspond to Kalman's alternate realizations of the same input-output relation. "Strongly connected" in Moore's terminology means completely controllable in Kalman's. "Indistinguishable states" in Kalman's terminology corresponds to states whose difference, not zero, is an observable state.

1.2 Invariance. It will be shown in this paper that the concept of controllability is closely related to the concept of invariance. The problem of the invariance in the theory of automatic control system reduces itself basically to defining conditions under which disturbances do not affect controlled variables. Although, as Minorsky [191] observed, the origin of this idea is not new, the interesting part of it lies in the attempts to reach a more general viewpoint on the whole subject.

To illustrate the concept of invariance, consider a certain dynamical system (electrical, mechanical, economic or some other) consisting of a number of subsystems or elements interconnected in some way. It is clear that such a system with, say, n degrees of freedom, when acted on by some excitation force $\underline{u}(t)$, will start oscillating, generally in a complicated manner, since the action of $\underline{u}(t)$ applied to one point of the system will be transmitted to all other parts through its various couplings. Suppose now we isolate as an abstraction a certain part of the system and inquire what conditions should be imposed on the parameters of the system so as to render this part immune against motions going on in other parts of the system under the influence of the applied force $\underline{u}(t)$. It is immaterial in this context whether this force is some control action or some disturbance. If the isolated part of the system is connected to the rest of the system, say, by connections A and B, necessary and sufficient conditions for that part to remain unaffected by processes going on in other parts of the system is that A and B were the nodes of the distribution of oscillations throughout the whole system.

Clearly, if these nodes exist at the points A and B, the portion of the system between A and B will not be acted on by any "external" forces and, from this point of view, will behave exactly in the same manner as if connections A and B were altogether removed.

Such a property of the subsystems as discussed in the above example is referred to as its invariance [191].

Most of the work on invariance problems and excitation control has been done in the Soviet Union. This activity remained practically unnoticed in the United States, even though invariance theory has many points of contact with research which has been done here under the name of sensitivity analysis of control systems, feedforward control, etc.

The first significant achievement in the invariance theory in the Soviet Union is credited to the work of Shchipanov in 1938 [250]. This work resulted from an attempt to develop an "ideal regulator" in which error would be reduced to zero. Although his original work contained some errors and misformulations, the idea of compensation for external effects as the operating principle of certain automatic control devices proved very fruitful. The term "invariance" as applied to this set of problems was first introduced by Luzin [173], who was also the one to develop precise mathematical formulation and theory. It was part of his more extensive study of the matrix theory of differential equations. He also studied the case when the disturbance effects can be made arbitrary small in systems where absolute invariance cannot be implemented. This is referred to in the literature as ε-invariance [174, 175, 176]. In the years 1945-1955 significant contributions to the basic research in this area were made by Kulebakin [148, 149, 150, 151, 153], who concentrated his attention on the feasibility of application of invariance theory to real life systems and on the development of invariant systems using both feedback and excitation control principles. The theory was further developed by Petrov [211, 212, 213, 214], and the so-called Kiev group which includes Ivakhnenko [111], Kukhtenko [143-147] and others. Ulanov proved the important theorem which says that conditions for minimizing control error are in many cases identical with invariance conditions [273, 274]. Taking a somewhat different approach then the matrix method introduced by Luzin, Rozonoer [232, 233] proved that invariance can be considered essentially a variational problem and, using variational approach, derived necessary and sufficient conditions of invariance for both linear and certain types of nonlinear dynamical systems.

The present state-of-the-art in invariance theory can be summed up briefly as follows:

(a) Theoretical foundations of invariant systems have been essentially developed;

(b) Criteria have been found for the physical realizability of the invariance principle;

(c) Various forms of invariance have been investigated and physically implemented;

(d) Relations between the conditions of invariance and the conditions of autonomy, optimal control, self-adaptive systems, etc. have been studied;

(e) Methods of extending the invariance principle to nonlinear systems have been discovered;

(f) It has been shown that theory of invariance is fundamental in developing the theory of accuracy and performance sensitivity of dynamic systems;

(g) Many real life applications of invariance theory have been successfully implemented in the design of control systems [16, 17, 26].

1.3 Interaction. The major result of this thesis is the demonstration of how controllability, observability, and invariance criteria can be applied in the analysis of interaction in systems as well as in the design of systems with desired interaction (i.e., non-interaction) characteristics.

Interaction is a problem of theoretical and practical interest in multivariable systems in general and multivariable control systems in particular. Although the theory of multivariable control has made some advances in recent years, it is still lagging behind the development of the theory of univariable control, i.e., control processes in which there is only one input to and one output from the system. In the case of a multivariable system, i.e., a system with several inputs and outputs, the classical design procedure is to select input-output pairs and to design conventional single variable systems neglecting the interaction or couplings which might exist among them. However, it was realized during the last two decades that consideration of system interactions leads to the design of higher performance control systems, i.e., systems with shorter response times and higher flexibility. The major thrust was made towards the design of so-called noninteracting controls, in which input-output pairs do not at all affect other input-output pairs, so that each output depends only on one pre-selected input and is independent of all other inputs.

In the literature concerned with the control of multivariable systems, the synthesis of noninteracting controllers has probably received more attention than any other aspect of this complex and frequently encountered design problem [209, 32, 46, 78, 193]. Even though it is true that noninteracting control, by breaking a complex dynamic system down into several isolated and simple systems, considerably simplifies the optimization procedures of complex structures, the limitations of this approach are also obvious. First of all, it is applicable only to systems with the same number of inputs and outputs. This severely restricts the class of systems to be considered. But more important than that, for most of the performance criteria and the constraints that are usually selected, a noninteracting system is generally not quite as good as an optimum interacting system [209].

A more fundamental approach to the interaction problems was recently taken by Mesarovic [187, 188], Narendra [195] and others. It was shown that noninteracting controls are only one aspect of interaction problems, which is termed input or cross-transfer interaction, and is roughly defined as the extent to which all inputs affect all outputs. Thus complete input noninteraction is the conversion of an n-input, n-output multivariable system into n single-variable inputs, in which each input affects one and only one output. Other types of interaction are "output interaction", which is basically the reaction of other outputs to an external disturbance applied to some output variable, and "output dependence", which refers to the possibility of obtaining any independent set of desired output functions by suitable manipulation of the input functions [187]. The studies of these more complicated aspects of interaction in systems are not too far advanced.

CHAPTER II

DYNAMICAL SYSTEMS

The axiomatic definition of a dynamical system, given below, is essentially that of Weiss and Kalman [286]. The main purpose of these axioms is to motivate the concept of state for describing physical systems, on which concept Newtonian mechanics as well as quantum mechanics is based.

Definition 1. A dynamical system is a mathematical structure denoted by the septet $(\Sigma, T, \Omega, U, \phi, Y, \psi)$ where:

(1) Σ is an abstract space called state space and T is a set of values of time at which the behavior of the system is defined. T is an ordered subset of the real numbers, with the usual ordering $>$ (or $<$). If $t_1, t_o \in T$, the statement $t_1 > t_o$ (or $t_1 < t_o$) will mean that t_1 is in the future (or in the past) with respect to t_o.

(2) Ω and U are abstract spaces with Ω being the set of functions of time $u{:}T \to U$ which represent the admissible inputs to the system.

(3) For any initial time $t \in T$, any initial state $\underline{x} \in \Sigma$, and any input $\underline{u} \in \Omega$ defined for $t \geqslant \tau$ (or $t \leqslant \tau$), states at other values of time of the system are determined by a given transition function $\underline{\phi}$: $\Omega \times T \times T \times \Sigma \to \Sigma$, which is written as $\underline{\phi}_u(t, \tau, x)$. This function has the following properties:

(a) $\underline{\phi}_u (\tau; \tau, \underline{x}) = \underline{x}$ for any $\underline{u} \in \Omega$, $\tau \in T$, $\underline{x} \in \Sigma$

(b) $\underline{\phi}_u (t; \tau, \underline{x})$ is defined only when $t \geqslant \tau$ (or $t \leqslant \tau$)

(c) $\Phi_u(t_2; t_o, x) = \Phi_u(t_2; t_1, \Phi_u(t_1; t_o, \underline{x})]$ for all $\underline{u} \in \Omega$, all t_o, t_1, t_2 in T such that $t_2 \geq t_1 \geq t_o$ (or $t_2 \leq t_1 \leq t_o$), and all $\underline{x} \in \Sigma$.

(d) If $\underline{u}_{[\tau, t]}$ denotes the equivalence class of functions in Ω whose values agree with $\underline{u}$ on the set $[\tau, t] \cap T$, then

$$\Phi_u(t; \tau, \underline{x}) = \Phi_{u_{[\tau, t]}}(t; \tau, \underline{x}).$$

(4) Every output of the system at time t is given by the value of a real function $\psi : T \times \Sigma \times \Omega \to R$, where ψ belongs to a given class Y.

(5) The functions ϕ and ψ are continuous with respect to suitable topologies defined on Σ, T, Ω, Y, and the reals, as well as the induced product topologies.

Dynamical systems can be causal, anticausal or noncausal. If the present values of the input, output, and state do not depend upon the future values of the input, output, and state, then we say that the system is <u>causal</u> or nonanticipatory. It is a system in which the time ordering is the usual one (>), and in which the state at the present summarizes the present and the past history of the system. An <u>anticausal</u> system is one in which the time ordering is the opposite of the usual one (<), and in which the state at the present summarizes the present and the future history of the system. A <u>noncausal</u> system is one which is neither causal nor anticausal. An example of such a system is the ideal low pass filter whose behavior at any given time depends on both past and future inputs [286, 291].

In the sequel we shall be concerned only with causal systems. Most of the conclusions pertaining to causal systems can be made to hold for anticausal systems simply by reversing the order of the time set [286]. We shall further assume that the quantities and functions with which we deal are exactly determined, so that no random elements occur. In other words, we shall suppose that our systems are completely deterministic and that our systems obey the classical cause-and-effect laws of physics.

A dynamical system is said to be continuous-time system if the set T, the domain of definition of the system, is an open interval (which may be all of R).

If Σ is assumed finite dimensional, then it can be proved [286] that the transition function of the most general dynamical continuous-time system is a solution of the vector differential equation

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{f}\,[\underline{x}(t), \underline{u}(t), t] \\ \underline{y}(t) &= \underline{g}\,[\underline{x}(t), \underline{u}(t), t] \end{aligned} \tag{2.1}$$

where

$\underline{x}(t)$ is an n-component state vector

$\underline{u}(t)$ is an r-component input vector

$\underline{y}(t)$ is an p-component output vector

$\underline{f}$: $R^n \times R^r \times T \to R^n$ is a continuous function of all its arguments and R^n and R^r are n- and r-dimensional real spaces, respectively

$\underline{g}$: $R^n \times R^r \times T \rightarrow R^p$ is a continuous function of all its arguments and $t \in T$, where T is the domain of the definition of the system.

The solution of (.1) can then be written

$$\underline{x}(t) = \underline{\phi}_u [t;\ t_o,\ \underline{x}_o]$$

with $\underline{\phi}_u$ as defined on pp. 16-17 and $\underline{x}_o = \underline{x}(t_o)$.

CHAPTER III

CONTROLLABILITY, OBSERVABILITY AND INVARIANCE DEFINITIONS

Besides stability, the properties of controllability, observability, and invariance are the most important structural properties of the system. We call them structural because they depend primarily on the structure of the system rather than inputs and outputs even though these properties might be manifest for only certain types of inputs or outputs (e.g., invariance of certain outputs of the system with respect to inputs representing periodic functions only).

Roughly speaking, we shall say that system is controllable if it is possible to drive any state (or output) of the system to the origin or to some other point in the state (or output) space in some finite time. We shall say that a system is observable if the initial state of the system can be found from a suitable measurement of the output. Finally, we shall say that a system is invariant to some input if the response (output) of the system does not depend on the particular input except possible for the initial conditions.

We shall show in the sequel that the properties of controllability, observability, and invariance can be uniquely related to interaction or noninteraction of system inputs and outputs, and can be used to define necessary and sufficient conditions for such interaction or noninteraction.

More precisely, consider the system Eq. (2.1) with an n-dimensional vector space of states, an r-dimensional vector space of inputs and p-dimensional output space. Let $\underline{x}(t_o) = \underline{x}_o$ be some initial state, and $t_o \varepsilon T$ some initial time. Let $x_j(t)$ $j = 1, 2, \ldots, n$, denote the j-th component of the state vector, $u_i(t)$ $i = 1, 2, \ldots, r$ the i-th component of the input vector, and $y_k(t)$, $k = 1, 2, \ldots, p$, the k-th component of the output vector. For this system, the concepts of state and output controllability, observability and state and output invariance are defined as follows.

3.1. <u>State Controllability</u>. Instead of the customary approach of considering a general case of a point in the phase or state space of the system and the conditions under which the system can be transferred to some other point in that space, we shall consider first only one component (coordinate) $x_i(t)$, $i = 1, 2, \ldots, n$ of the state vector $\underline{x}(t)$ and one component $u_j(t)$ of the input vector $\underline{u}(t)$, and define <u>selective state controllability</u>. Thus we shall say:*

<u>SC (I, 1)</u>. The i-th component x_{io} **of the state** $\underline{x}_o$ is <u>selectively controllable</u> by the j-th component $u_j(t)$ of the input $\underline{u}(t)$ if there exists some finite time $t_1 > t_o$ and some input $u_j[t_o, t_1]$ which transfers (x_{io}, t_o) to (x_{i1}, t_1)**, where $u_{j[t_o, t_1]}$ is the j-th component of the input $\underline{u} \varepsilon \Omega$ and $u_k = 0$ for $k \neq i$, $k, i = 1, 2, \ldots, r$.

<u>SC (I, 4)</u>. If every point x_{io} in the i-th coordinate set

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* The definitions are assigned codes of the Type SC (,) to facilitate identification of corresponding entrances in Table 1 of the Appendix, and elsewhere in the text. See page 24 for further details.

** Note that the couples (x_{io}, t_o) and (x_{i1}, t_1) are projections of the phases $(\underline{x}_o, t_o)$ and $(\underline{x}_1, t_1)$ on the i-th coordinate axis.

Σ_i, i = 1, 2, ..., n, is controllable at t_o, then the system is selectively i-th state controllable at t_o.

SC (I, 44). If every point $x_{io} \varepsilon \Sigma_i$, i = 1, 2, ..., n is controllable at every $t_o \varepsilon$ T, where T is the interval of the definition of the system then the system is selectively completely i-th state controllable.

SC (III, 1). The i-th component x_{io} of the state $\underline{x}_o$ is strongly controllable at t_o if it is selectively controllable by all input components $u_{j[t_o, t_1]}$, j = 1, 2, ..., r, separately.

SC (III, 4). If every point $x_{io} \varepsilon \Sigma_i$ is controllable, then the system is strongly i-th state controllable system at t_o.

SC (III, 44). If, in addition, this is true for all $t_o \varepsilon$ T, then the system is strongly completely i-th state controllable.

SC (I, 444). If a system is selectively completely i-th state controllable on every interval $[t_o, t_1] \subset$ T, then it is selectively totally i-th state controllable.

SC (III, 444). Combining the definitions SC (III, 1) and SC (I, 444), we obtain valid cases of a system which is strongly totally i-th state controllable.

SC (I, 4444). If a system is selectively totally i-th state controllable on any region of the (t, τ) plane, then the system is said to be selectively uniformly i-th state controllable.

SC (III, 1111). If a system is strongly totally i-th state controllable on any region of the (t, τ) plane, then the system is said to be i-th state-normal.

SC (I, 2). If more than one component of the state $\underline{x}_o$ are controllable by the j-th component of the input $\underline{u}$ (t), j = 1, 2, ..., r, in the sense that there exists some finite time $t_1 > t_o$ and some input $u_{j[t_o, t_1]}$ which transfers $(\tilde{\underline{x}}_o, t_o)$ to $(\tilde{\underline{x}}_1, t_1)$, where $\tilde{\underline{x}}_o, \tilde{\underline{x}}_1$ are points in the space $\tilde{\Sigma} \subset \Sigma$ of the dimensionality corresponding to the number of controllable components of $\underline{x}_o$, then we shall say that $\underline{x}_o$ is <u>selectively controllable state in the space $\tilde{\Sigma} \subset \Sigma$</u>.

As in the case of a single component of the state, we can define complete, strong, total etc., controllability in the space $\tilde{\Sigma} \subset \Sigma$. Alternatively, we shall say that the system is controllable in the subset $\tilde{\underline{X}}$ of the set $\overline{\underline{X}}$ (of the components of state vector).

Generalization of the concept of selective state controllability leads to the concept of j-th input controllable states.

<u>SC (I, 3)</u>. A state $\underline{x}_o \in \Sigma$ is said to be <u>selectively controllable</u> if there exists some finite $t_1 > t_o$ and some input $u_{j[t_o, t_1]}$, j = 1, 2, ..., r, representing the j-th component of $\underline{u} \in U$, which transfers $(\underline{x}_o, t_o)$ to $(\underline{x}_1, t_1)$ where $u_{k[t_o, t_1]} = 0$ for $k \neq j$.

<u>SC (I, 6)</u>. If every state $\underline{x}_o \in \Sigma$ is selectively controllable, then we have a <u>selectively state controllable system at</u> t_o (by the j-th input).

<u>SC (I, 66)</u>. If every state is j-th input controllable for all $\underline{x}_o \in \Sigma$ and $t_o \in T$, then the system is <u>selectively completely state controllable</u>.

<u>SC (III, 6666)</u>. If the system is j-th input controllable separately for all $u_{j[t_o, t_1]}$, j = 1, 2, ..., r, and on all regions of the (t,τ) plane, then the system is called <u>state-normal</u> [15].

If we now consider the overall $\underline{u}_{[t_o, t_1]}$, then we are lead to state-controllability definitions in the current control theory literature. We state some of these definitions below.

SC (II, 3). The state $\underline{x}_o \in \Sigma$ is controllable at time t_o if there exists an input function $\underline{u}_{[t_o, t_1]}$, depending in general on $\underline{x}_o$ as well as t_o, which transfers $(\underline{x}_o, t_o)$ to $(\underline{x}_1, t_1)$ in finite time $t_1 > t_o$.

SC (II, 6). If every state $\underline{x}_o \in \Sigma$ is controllable at time $t_o \in T$, then the system is state-controllable at t_o.

SC (II, 66). If the system is state-controllable in the above sense for all $t_o \in T$, then it is completely state-controllable.

It is further possible to define strongly and totally controllable states and systems as well as state-proper and state-normal cases [142, 162, 284].

A complete listing of various state-controllability concepts based on initial state and time variations and criteria for their identification are given in a condensed form in Appendix Table 1. It should be noted that the listing is not considered exhaustive, since there are still other types of controllability, such as differential controllability, ε-controllability etc., which have not been considered. This remark applies also to observability and invariance sections, which are dealt with in the sequel.

For easy reference, various types of state controllability listed in Table 1 of the Appendix have been assigned identification codes consisting of letters SC, which stand for state controllability, followed by a Roman and an Arabic number in parenthesis. The Roman numbers identify controlling inputs as follows:

I - controllable by some component $u_{j[t_o,\ t_1]}$ of the input $\underline{u}_{[t_o,\ t_1]}$.

II - controllable by the input $\underline{u}_{[t_o,\ t_1]}$ as a whole.

III - controllable by each of the components $u_{j[t_o,\ t_1]}$, $j = 1, 2, \ldots, k$, of the input $\underline{u}_{[t_o,\ t_1]}$ separately.

The Arabic numbers are either one, two, three, or four digits long. Digits 1 through 6 have been used to indicate the type of controlled variables as follows (in conjunction with control category I, II, or III as defined above):

1 - some component x_i (t) of the state vector $\underline{x}$ (t) is controllable for some initial condition x_{io}

2 - several components of the state vector $\underline{x}$ (t), constituting the projection $\underline{\tilde{x}}$ (t) on the subspace $\tilde{\Sigma}$ of the state space Σ, are simultaneously controllable for <u>some</u> initial conditions $\underline{\tilde{x}}_{io}$

3 - the state $\underline{x}$ (t) is controllable for <u>some</u> initial value $\underline{x}_o$.

4 - some component x_i (t) of the state vector $\underline{x}$ (t) is controllable for <u>all</u> initial values x_{io} .

5 - several components $\underline{\tilde{x}}$ (t) of the state vector $\underline{x}$ (t) are simultaneously controllable for <u>all</u> initial conditions $\underline{\tilde{x}}_{io}$.

6 - the state $\underline{x}$ (t) is controllable for <u>all</u> initial values $\underline{x}_o$.

The number of digits in this portion of the code indicates the time interval of controllability in the following manner:

one digit - controllable for <u>some</u> initial value $t_o \in T$

two digits - controllable for <u>all</u> initial values $t_o \in T$

three digits - controllable on <u>all</u> intervals $[t_o, t_1] \subset T$

four digits - controllable for all $(t, \tau) \in T \times T$, i.e., on the whole $T \times T$ plane.

As it will be shown later, this method of denoting various types of state controllability (as well as output controllability, observability, and state and output invariance) was selected for the purpose of notationally indicating certain basic implications among various types of controllability (observability, invariance).

3.2. <u>Output Controllability</u>. Essentially, output controllability means that any final output of the system can be attained, starting with given or arbitrary initial conditions and, as in the case of state controllability, various types of output controllability are defined depending on whether this capability is limited to certain initial conditions or valid for all initial conditions, and whether it is limited to certain initial times, or valid for all initial times or even all time intervals on the interval of definition of the system. It is a simple matter to relate output controllability to state controllability, as it will be shown in the next paragraph. As a matter of fact, state controllability can be considered as a special case of output controllability, though for linear systems only.

The definitions of various types of output controllability are basically the same as those of state controllability, with "state" replaced by "output" and $\underline{x}$'s replaced by $\underline{y}$'s, with y_i (t), i = 1, 2, ..., p, denoting the i-th component of the output vector $\underline{y}$ (t) .

Table 2 of the Appendix lists various types of output controllability in the same order as various types of state controllability were listed in Table 1 of the Appendix. They are also coded identically except that the letters SC are replaced by the letters OC to indicate output controllability.

3.3 Observability. As in the preceding section on controllability, we shall define various types of observability in greater detail than it has been done so far in control theory literature. Thus, we shall start with the definitions of "selective" observability relating only certain components of output and state vectors and then consider more general cases. The sequence of the definitions is the same as for state controllability and the definitions of controllability and observability are equivalent in the sense that for each controllability definition there is a counterpart in observability definition. All the definitions refer to the system specified by the equations (2.1).

OB (I, 1). The component x_{io} of the state $\underline{x}_o$ is selectively-observable i-th state component at time t_o if there exists a time $t_1 > t_o$ such that x_{io} can be determined from knowledge of the input $\underline{u}_{[t_o, t_1]}$ and the component $\underline{y}_{j[t_o, t_1]}$ of the systems output $\underline{y}$ (t) on the interval $[t_o, t_1]$.

OB (I, 4). If every point x_{io} in the coordinate set Σ_i is observable at t_o, then we have a selectively i-th state observable at t_o.

OB (I, 44). If every point $x_{io} \varepsilon \Sigma_i$, is observable at every t_o

in the interval of definition of the system T, then the system is <u>selectively completely i-th state observable</u> (with respect to some output component $y_{j[t_o, t_1]}$).

<u>OB (III, 1)</u>. The component x_{io} of the state $\underline{x}_o$ is <u>strongly observable i-th state component at t_o</u> if it is selectively observable with respect to each output components $y_{j[t_o, t_1]}$, $j = 1, 2, \ldots, p$, separately.

<u>OB (III, 4)</u>. If every point $x_{io} \in \Sigma_i$ is observable, then it is a <u>strongly i-th state observable system at t_o</u>.

<u>OB (III, 44)</u>. If, in addition, this is true for all $t_o \in T$, then the system is <u>strongly completely i-th state observable</u>.

<u>OB (I, 444)</u>. If the i-th component x_{io} of the state $\underline{x}_o$ is completely observable on every interval $[t_o, t_1] \subset T$, for all $x_{io} \in \Sigma_i$ then it is <u>selectively totally i-th state observable system</u>.

Systems, which are totally observable with respect to each output component $y_{j[t_o, t_1]}$, $j = 1, 2, \ldots, p$, are <u>strongly totally observable</u>, so that, depending on the number of observable components, we specifically can have:

<u>OB (III, 444)</u> - a <u>strongly totally i-th state observable system</u>

<u>OB (III, 555)</u> - a <u>strongly totally observable system in $\tilde{\Sigma} \subset \Sigma$</u>,

<u>OB (III, 666)</u> - <u>a strongly totally observable system</u>.

<u>OB (I, 4444)</u>. If a system is selectively totally i-th state observable on any region of the (t, τ) plane, then the system is said to be <u>selectively uniformly i-th state observable</u>.

<u>OB (III, 4444)</u>. If the i-th component x_{io} of the state $\underline{x}_o$, is

strongly totally observable on any region of the (t, τ) plane, then the system is said to be <u>i-th state observable normal system</u>.

<u>OB(I, 2)</u>. If a state $\underline{x}_o$ of the system is observable in more than one of its components with respect to the j-th output component (in the sense that $\tilde{\underline{x}}_o$, which are points in the subspace $\tilde{\Sigma}$ of the space Σ of the dimensionality corresponding to the number of observable coordinates of $\underline{x}_o$), then we shall say that the state is <u>selectively observable in the space</u> $\tilde{\Sigma}$.

We can further define complete, strong, total etc., observability in the subspace $\tilde{\Sigma}$ of Σ.

Requirement that all components of the state vector be simultaneously observable with respect to a certain component of the input vector $\underline{y}(t)$ leads to observability definitions:

<u>OB(I, 3)</u>. A state $\underline{x}_o \in \Sigma$ is said to be <u>selectively observable state at time</u> t_o if there exists some finite $t_1 > t_o$ such that $\underline{x}_o$ can be determined from knowledge of the input $\underline{u}_{[t_o, t_1]}$ and the j-th component of the systems output $y_{j[t_o, t_1]}$.

<u>OB(I, 6)</u>. If every state $\underline{x}_o \in \Sigma$ is selectively observable at t_o, then the system is <u>a selectively observable system at</u> t_o.

<u>OB(I, 66)</u>. If every state $\underline{x}_o \in \Sigma$ is selectively observable for all $\underline{x}_o \in \Sigma$ and all $t_o \in T$, then the system is <u>completely selectively observable</u>.

Again, if we consider the overall output $\underline{y}_{[t_o, t_1]}$ instead of its single components, we come up with the customary definitions of observability, of which only the three most important ones are given below (definitions of other types of observability can be easily constructed from Appendix Table 3).

OB (II, 3). A state x is observable at time t_o if there is a time $t_1 > t_o$ such that knowledge of the input $\underline{u}_{[t_o, t_1]}$ and system's output $\underline{y}_{[t_o, t_1]}$ is sufficient to determine $\underline{x}_o$.

OB (II, 6). If every state $\underline{x}_o \in \Sigma$ is observable at time $t_o \in T$, then the system is an observable system at t_o.

OB (II, 66). If the system is observable for all $t_o \in T$, then it is (completely) observable.

Table 3 in the Appendix gives a condensed representation of various types of observability, of which only the more important ones were explicitly defined in the preceding paragraphs. These types are numbered in the same order as various types of controllability were numbered in Tables 1 and 2, except that the letters OB are used in front of the number to indicate that it pertains to certain observability definition.

3.4. State Invariance. Invariance, either state or output, is the third basic property of a system to be defined and later investigated in the context of interactions of system's inputs and outputs. The definitions follow the pattern established for controllability and observability and refer to the system represented by the equations (2.1).

SI (I, 1). The component $x_i(t)$ of the state vector $\underline{x}(t)$ is selectively invariant i-th state component at time t_o with respect to the j-th input component $u_j(t)$ if there exists $t_1 > t_o$ such that the values of $x_i(t)$ on $[t_o, t_1]$ do not depend on the values of $u_{j[t_o, t_1]}$ on the interval $[t_o, t_1]$.

SI (I, 4). If every point x_{io} in the coordinate set Σ_i is selectively invariant at time $t_o \in T$, then the system is selectively j-th input state-invariant at t_o (with respect to some j-th input component

u_j (t)).

If for some $x_i(t_o) = x_{io}$ the i-th component $x_i(t)$ of the state vector $\underline{x}(t)$ is selectively invariant (with respect to the j-th input component u_j) for all $t_1 \varepsilon T$ then the system is a <u>selectively completely i-th state-invariant system</u>.

<u>SI (III, 1)</u>. The state component $x_i(t)$ of $i = 1, 2, \ldots, n$ the state vector $\underline{x}(t)$ is <u>strongly invariant at time</u> $t_o \varepsilon T$ if it is selectively invariant with respect to all input components $u_j(t)$, $j = 1, 2, \ldots, r$.

<u>SI (III, 4)</u>. If this is true for every point $x_i(t_o) = x_{io} \varepsilon \Sigma$, then the system is <u>strongly i-th state-invariant at</u> t_o.

<u>SI (III, 44)</u>. Furthermore, if this is true for all $t_o \varepsilon T$, then the system is <u>strongly completely i-th state-invariant</u>.

<u>SI (I, 444)</u>. If the i-th component $x_i(t)$ of the state $\underline{x}(t)$ is selectively completely invariant on every interval $[t_o, t_1]$, $t_o, t_1 \varepsilon T$ then we say that it is a <u>selectively totally i-th state-invariant system</u>.

<u>SI (III, 444)</u>. If the component $x_i(t)$ is selectively totally i-th state-invariant for all inputs $u_{j[t_o, t_1]}$, $j = 1, 2, \ldots, r$, then it is <u>strongly totally i-th state-invariant system</u>.

<u>SI (I, 4444)</u>. If the component $x_i(t)$ is selectively totally i-th state-invariant in any region of the (t, τ) plane then it is a <u>selectively uniformly i-th state-invariant system</u>.

<u>SI (III, 4444)</u>. If the component $x_i(t)$ is strongly totally i-th state-invariant on any region of the (t, τ) plane, then it is said to be <u>an i-th state-invariant normal system</u>.

<u>SI (I, 2)</u>. If more than one of the components of the state $\underline{x}(t)$

are selectively invariant with respect to some input component u_j (t) in the sense that the values of $\tilde{\underline{x}}$ (t), where $\tilde{\underline{x}}$ (t) is the trajectory in the subspace $\tilde{\Sigma}$ of Σ of the dimensionality corresponding to the number of invariant components of $\underline{x}$ (t), are not affected by the input u_j (t) on $[t_o, t_1]$, then we say that it is a <u>selectively invariant state in the subspace $\tilde{\Sigma}$</u>.

<u>SI (I, 5)</u>. If this is true for all initial values $\tilde{\underline{x}}_o$ (t_o) = $\tilde{\underline{x}}_{io}$, then the system is <u>selectively state-invariant in the subspace $\tilde{\Sigma}$ at t_o</u>.

<u>SI (I, 55)</u>. If this holds for all times $t_o \varepsilon$ T, then the system is <u>selectively completely state-invariant in $\tilde{\Sigma}$</u>.

Again, we can generalize the concept of invariance by requiring that all components of a state vector be invariant with respect to a certain input component. The respective definitions are:

<u>SI (I, 3)</u>. The state $\underline{x}$ (t_o) = $\underline{x}_o$ is <u>selectively invariant at time</u> $t_o \varepsilon$ T (with respect to some partial input u_j (t)) if there exists $t_1 > t_o$ such that the values $\underline{x}$ (t) on $[t_o, t_1]$ do not depend on the values of the j-th input component u_j (t) on the interval $[t_o, t_1]$.

<u>SI (I, 6)</u>. If every state $\underline{x}$ (t_o) is selectively invariant at time $t_o \varepsilon$ T, then it is a <u>selectively state-invariant system at</u> t_o.

<u>SI (I, 66)</u>. If the system is selectively state-invariant for all $t_o \varepsilon$ T, then we say that the system is <u>selectively completely state-invariant</u>.

Further generalization leads to the following definitions:

SI(II, 3). The state $\underline{x}(t_o) = \underline{x}_o$ is invariant at time $t_o \in T$ if there exists $t_1 > t_o$ such that the values $\underline{x}(t)$ on the interval $[t_o, t_1]$ do not depend on the values $\underline{u}(t)$ on the same interval.

SI(II, 6). If every state $\underline{x}(t_o) = \underline{x}_o$ is invariant at time $t_o \in T$, then the system is state-invariant at t_o.

SI(II, 66). If the system is state-invariant for all $\underline{x}_o$ and all times $t_o \in T$, then we say that it is a (completely) state-invariant system.

As was done in the sections on controllability and observability, it is further possible to define strongly and totally invariant states and systems, etc.

3.5. Output Invariance. Invariance in state space does not necessarily entail invariance in the output space with respect to certain inputs. For plants with direct transmission, i.e., for systems (2.1) with $G \neq 0$, this is self evident. However, it can be easily ascertained that this is also true for systems without direct transmission. In general, state invariance can be considered as a special case of output invariance, similarly as state controllability can be viewed as a special case of output controllability.

Definitions of output invariance are identical with those of state invariance except that the term "state" is replaced by "output" and the variables $x_i(t)$ or $\underline{x}(t)$ are replaced by the variables $y_i(t)$ or $y(t)$, as the need might be. For this reason it was not deemed necessary to repeat the definitions in full. Instead, reference is made to Table 5 in the Appendix which lists various types of output invariance in the same order as Table 4 in the Appendix does for state invariance. The letters

OI are used in the code to indicate output invariance.

CHAPTER IV

BASIC IMPLICATIONS

Within each category discussed in the preceding paragraphs, certain types of controllability, observability or invariance, which are of more general nature, imply certain specific types of controllability, observability or invariance. Implications, which are deducible from the definitions and are valid for all types of systems, will be referred to as basic implications. For example, if a system is completely state controllable, then it must be a state controllable system at t_o as well, i.e.,

$$SC(II, 33) \Rightarrow SC(II, 3)$$

Similarly, selectively totally i-th state observable system implies selectively observable i-th state component for all $[t_o, t_1] \subset T$, or

$$OB(I, 444) \Rightarrow OB(I, III)$$

where SC(II, 3), SC(II, 33), OB(I, 111), and OB(I, 444) are the identification codes of the above types of controllability and observability from Appendix Tables 1 and 3.

We recall that in general the Roman numerals in the identification codes have the following meaning:

I - selective [controllability, observability or invariance]

II - (weak) [controllability, observability or invariance]

III - strong [controllability, observability or invariance]

It can be easily verified that the following chain of basic implications holds with respect to the above three properties.

$$XX(III, xx) \Rightarrow XX(II, xx) \Rightarrow XX(I, xx)$$

where capital XX are to be replaced by SC (state controllability), OC (output controllability), OB (observability), SI (state invariance), or OI (output invariance), and small xx in the brackets are to be replaced by one of the valid arabic numbers appearing in the identification codes. For example

$$OC(III, 444) \Rightarrow OC(II, 444) \Rightarrow OC(I, 444)$$

means that strongly totally i-th output controllable system implies totally i-th output controllable system which in turn implies selectively totally i-th output controllable system.

Within each of the categories of selective (I), weak (II), and strong (III) controllability, observability, or invariance properties, we can identify an ordered set of basic implications resulting from the fact that time sets and state or output spaces in the definitions can be ordered by inclusion relation. Thus, on one hand, we have a set of implications

$$\begin{array}{lllllll}
XX(\cdot,\ 1111) & \Rightarrow & XX(\cdot,\ 111) & \Rightarrow & XX(\cdot,\ 11) & \Rightarrow & XX(\cdot,\ 1) \\
XX(\cdot,\ 2222) & \Rightarrow & XX(\cdot,\ 222) & \Rightarrow & XX(\cdot,\ 22) & \Rightarrow & XX(\cdot,\ 2) \\
XX(\cdot,\ 3333) & \Rightarrow & XX(\cdot,\ 333) & \Rightarrow & XX(\cdot,\ 33) & \Rightarrow & XX(\cdot,\ 3) \\
XX(\cdot,\ 4444) & \Rightarrow & XX(\cdot,\ 444) & \Rightarrow & XX(\cdot,\ 44) & \Rightarrow & XX(\cdot,\ 4) \\
XX(\cdot,\ 5555) & \Rightarrow & XX(\cdot,\ 555) & \Rightarrow & XX(\cdot,\ 55) & \Rightarrow & XX(\cdot,\ 5) \\
XX(\cdot,\ 6666) & \Rightarrow & XX(\cdot,\ 666) & \Rightarrow & XX(\cdot,\ 66) & \Rightarrow & XX(\cdot,\ 6)
\end{array}$$

On the other hand, we have sets of implications of the type shown below, which hold for one, two, three and four digit code groups in exactly the same fashion as for the one digit, i.e.,

$$\begin{array}{ccc}
XX(\cdot,\ 6) & \Rightarrow & XX(\cdot,\ 3) \\
\Downarrow & & \Downarrow \\
XX(.,\ 5) & \Rightarrow & XX(\cdot,\ 2) \\
\Downarrow & & \Downarrow \\
XX(\cdot,\ 4) & \Rightarrow & XX(\cdot,\ 1)
\end{array}$$

A complete structure of implications relating definitions within each category - selective (I), weak (II), and strong (III) - of controllability, observability or invariance is shown in the Appendix, Table 6.

There are few more implications which do not have universal validity for all properties, but only for some. For instance, a system is totally state controllable only if it is a state-proper system. This is in contrast with output-controllability where a system may be totally output controllable without being necessarily output-proper [142].

CHAPTER V

LINEAR DYNAMICAL SYSTEMS

To prevent the main course of this presentation from being obscured by excessive detail we shall limit our investigation to linear continuous-time dynamical systems. In other words, we shall restrict our attention to a special class of dynamical systems Eq. (2.1) which are:

(1) finite-dimensional (Σ).

(2) continuous-time (T is a subset of real numbers, and ϕ, ψ are smooth real functions of t).

(3) linear (ψ is linear in $\underline{x}$ and ϕ is linear jointly in $\underline{x}$ and $\underline{u}$)

(4) multi-input, multi-output (U is r-dimensional, and Y is p-dimensional linear vector spaces).

The transition function of the most general dynamical system which satisfies the axioms of the Definition 1 and the above conditions, is a solution of the vector differential equation

$$\begin{aligned} \dot{\underline{x}}(t) &= A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) \\ \underline{y}(t) &= F(t)\,\underline{x}(t) + G(t)\,\underline{u}(t) \end{aligned} \tag{5.1}$$

where

$\underline{x}(t)$ is an n-component state vector

$\underline{u}(t)$ is a r-component input vector

$\underline{y}(t)$ is an p-component output vector

$A(t)$ is a real n x n matrix

$B(t)$ is a real n x r matrix

$F(t)$ is a real p x n matrix

$G(t)$ is a real p x r matrix

For the most part, we shall consider, as a further simplification, linear time-invariant system given by the state-variable differential equation

$$\begin{aligned} \dot{\underline{x}}(t) &= A\,\underline{x}(t) + B\,\underline{u}(t) \\ \underline{y}(t) &= F\,\underline{x}(t) + G\,\underline{u}(t) \end{aligned} \tag{5.2}$$

with variables being defined as in Eq. (5.1), with the only difference that the matrices A, B, F, and G have constant elements. We shall show later that the extension of the main results to time-varying systems is straightforward.

Essentially we have two types of inputs to the system: manipulative or control inputs and disturbance inputs. Here, both control and disturbance inputs are assumed to be deterministic, even though the form of the disturbance inputs might not be known. Furthermore, for the purpose of this investigation, it is in most cases immaterial whether a particular input is a control input or a disturbance input. If we denote

$\underline{u}_c(t)$ = control or manipulative input

$\underline{u}_d(t)$ = disturbance input then

$$\underline{u}(t) = \begin{bmatrix} \underline{u}_c(t) \\ \underline{u}_d(t) \end{bmatrix}$$

represents all the inputs to the system. Thus, by appropriately partitioning the matrices B and G, we can write the system Eq. (5.2) in the form

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B_c\,\underline{u}_c(t) + B_d\,\underline{u}_d(t)$$

$$\underline{y}(t) = F\,\underline{x}(t) + G_c\,\underline{u}_c(t) + G_d\,\underline{u}_d(t)$$

On the other hand, every system of the type

$$\dot{x}(t) = A\,\underline{x}(t) + C\,\underline{u}_c(t) + D\,\underline{u}_g(t)$$

$$\underline{y}(t) = F\,\underline{x}(t) + H\,\underline{u}_c(t) + K\,\underline{u}_d(t)$$

can be represented in the form (5.2) with

$$B = [C \,\vdots\, D]\ , \quad G = [H \,\vdots\, K]$$

and

$$\underline{u}(t) = \begin{bmatrix} \underline{u}_c(t) \\ \underline{u}_d(t) \end{bmatrix}$$

Hence, for compactness and convenience of notation, we shall normally refer to the system equations (5.2) with the tacit assumption that

$\underline{u}(t)$ represents both control and disturbance inputs, and we shall interpret these inputs as the need may be.

CHAPTER VI

CONTROLLABILITY, OBSERVABILITY AND INVARIANCE CRITERIA FOR LINEAR TIME-INVARIANT DYNAMICAL SYSTEMS

Many of the fine differences which are valid for linear dynamical time-variant systems, disappear for time-invariant systems. This results in considerable reduction in the variety of types of controllability, observability, and invariance on the one hand, and in the simplification of criteria on the other.

First, if a linear time-invariant system (5.2) is controllable at some time $t_o \in T$, then it is controllable at any other time $t_o^* \in T$ [15]. Hence we can choose for convenience $t_o = 0$. A further consequence of this is the fact that if a linear time-invariant system is controllable (or observable, or invariant) at some time $t_o \in T$, then it is controllable (observable, invariant) on all intervals $[t_o, t_1]$, $t_1 > t_o$, t_o, $t_1 \in T$, and all regions $(t, \tau) \in T \times T$. In other words if, for example, a system is selectively i-th state controllable at t_o, then it is also selectively completely i-th state controllable, selectively totally i-th state controllable, and selectively uniformly i-th state controllable.

Another important property of linear time-invariant systems is that the set of states $\underline{x}_o$ which are controllable is a subspace of Σ [15]. In other words, if the component $x_i(t)$ of the state vector of the system (5.2) is controllable for some initial value x_{io}, then it is controllable for all initial values x_{io} in Σ_i. Thus, selectively uniformly controllable i-th state component implies (and is implied by) selectively uniformly i-th state controllable system [or SC(I, 1111) $\Longleftrightarrow$ SC(I, 4444)], a proper i-th state component implies i-th state proper system [SC(II, 1111) $\Leftrightarrow$ SC(II, 4444)]; and normal i-th state component implies i-th state-normal system [SC(III, 1111) $\Leftrightarrow$ SC(III, 4444)]. Similarly, if some components of the state vector $\underline{x}(t)$ are simultaneously controllable in some subspace $\tilde{\Sigma}$ of the state space Σ for some initial value $\underline{\tilde{x}}_o \varepsilon \tilde{\Sigma}$ then they are controllable for all initial values $\underline{\tilde{x}}_o \varepsilon \tilde{\Sigma}$. Hence, selectively uniformly controllable state in $\tilde{\Sigma}$ implies selectively uniformly state-controllable system in $\tilde{\Sigma}$ [SC(I, 2222) $\Leftrightarrow$ SC(I, 5555)], a proper state in $\tilde{\Sigma}$ implies state proper system in $\tilde{\Sigma}$ [SC(II, 2222) $\Leftrightarrow$ SC(II, 5555)] and normal state in $\tilde{\Sigma}$ implies state-normal system in $\tilde{\Sigma}$ [SC(III, 2222) $\Leftrightarrow$ SC(III, 5555)]. Finally, if some initial state $\underline{x}_o$ is controllable in Σ, then all states $\underline{x}(t)$ $\varepsilon \Sigma$ are controllable in Σ. Specifically, this means that a selectively uniformly controllable state implies selectively uniformly state controllable system [SC(I, 3333) $\Leftrightarrow$ SC(I, 6666)]; proper state implies state-proper system [SC(II, 333) $\Leftrightarrow$ SC(II, 6666)]; and normal state implies state-normal system [SC(III, 3333) $\Leftrightarrow$ SC(III, 6666)].

Hence, the number of different types of state controllability is considerably reduced for linear time-invariant systems. These types are presented concisely in Table 1a. A diagram of basic implications is shown in Table 1b.

It is easy to verify that the situation is quite similar with respect to output controllability, observability, state invariance and output invariance.

There is only one difference: a system is i-th state invariant or proper invariant with respect to the overall input $\underline{u}_{[t_o, t_1]}$ if and only if it is invariant with respect to each of the input components $u_{j[t_o, t_1]}$, $j = 1, 2, \ldots, r$, separately. Hence, proper i-th state invariant system implies normal i-th state invariant system [$\overline{SI}$(II, 4444) $\Longleftrightarrow$ $\overline{SI}$(III, 4444)]; proper state invariant system in $\tilde{\Sigma}$ implies normal state invariant system in $\tilde{\Sigma}$ [$\overline{SI}$(II, 5555) $\Longleftrightarrow$ SI(III, 5555)]; and proper state invariant system implies normal state invariant system [$\overline{SI}$(II, 6666) $\Longleftrightarrow$ $\overline{SI}$(III, 6666)]. This implication will follow immediately from the necessary and sufficient conditions for state invariance discussed later in this chapter. The above statement holds for output invariance as well. However, it does not hold for state or output controllability, and for observability.

The distinct types of output controllability, observability and state and output invariance of linear time-invariant systems are presented in Tables 2a through 5a, with implication structures shown in Tables 2b through 5b.

6.1. Necessary and sufficient conditions for state controllability. We shall now present the theorems which state necessary and sufficient conditions for state controllability of linear time-invariant systems. Proofs which are not readily available in the existing subject literature, will be given in full. Otherwise references will be made to sources where proofs can be found, and the proofs will not be repeated here.

First, we shall introduce some further notation. If A is a $m \times n$ matrix with constant elements, we shall denote by

$$\underline{a}_{i\cdot} = [a_{i1}, a_{i2}, \ldots a_{in}], \quad i = 1, 2, \ldots, m$$

Table 1a. Distinct Types of State Controllability of Linear Time-Invariant Dyanmical Systems

Control Input / Controlled Variable	$u_j[t_o, t_1]$ $\exists j = 1, 2, \ldots, r$	$\underline{u}[t_o, t_1]$	$u_j[t_o, t_1]$ $\forall j = 1, 2, \ldots, r$
$\forall x_{io} \varepsilon \Sigma_i$ $\exists i = 1, 2, \ldots, n$	$\overline{SC}$(I, 4444) Selectively uniformly i-th state-controllable system	$\overline{SC}$(II, 4444) Proper i-th state component	$\overline{SC}$(III, 4444) i-th state-normal system
$\forall \underline{\tilde{x}}_o \varepsilon \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$	$\overline{SC}$(I, 5555) Selectively uniformly state-controllable system in $\tilde{\Sigma}$	$\overline{SC}$(II, 5555) State-proper system in $\tilde{\Sigma}$	$\overline{SC}$(III, 5555) State-normal system in $\tilde{\Sigma}$
$\forall \underline{x}_o \varepsilon \Sigma$	$\overline{SC}$(I, 6666) Selectively uniformly state-controllable system	$\overline{SC}$(II, 6666) State-proper system	$\overline{SC}$(III, 6666) State-normal system

Table 1b. Structure of Basic Implications of State Controllability Types of Linear Time-Invariant Dynamical Systems

$$
\begin{array}{ccccc}
\overline{SC}(III, 6666) & \Rightarrow & \overline{SC}(II, 6666) & \Rightarrow & \overline{SC}(I, 6666) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\overline{SC}(III, 5555) & \Rightarrow & \overline{SC}(II, 5555) & \Rightarrow & \overline{SC}(I, 5555) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\overline{SC}(III, 4444) & \Rightarrow & \overline{SC}(II, 4444) & \Rightarrow & \overline{SC}(I, 4444)
\end{array}
$$

Table 2a. Distinct Types of Output Controllability of Linear Time-Invariant Dynamical Systems

Controlled Variable \ Control Input	$u_j[t_o, t_1]$ $\exists j = 1, 2, \ldots, r$	$\underline{u}[t_o, t_1]$	$u_j[t_o, t_1]$ $\forall j = 1, 2, \ldots, r$
$\forall y_{io} \in R$ $\exists i = 1, 2, \ldots, p$	$\overline{OC}$(I, 4444) Selectively uniformly i-th output controllable system	$\overline{OC}$(II, 4444) Proper i-th output component	$\overline{OC}$(III, 4444) i-th output-normal system
$\forall \tilde{\underline{y}}_{io} \in R^k$ $R^k \subset R^p, 1 < k < p$	$\overline{OC}$(I, 5555) Selectively uniformly output-controllable system in R^k, $1 < k < p$	$\overline{OC}$(II, 5555) Output-proper system in R^k $1 < k < p$	$\overline{OC}$(III, 5555) Output-normal system in R^k, $1 < k < p$
$\forall \underline{y}_o \in R^p$	$\overline{OC}$(I, 666) Selectively uniformly output-controllable system	$\overline{OC}$(II, 6666) Output-proper system	$\overline{OC}$(III, 6666) Output-normal system

Table 2b. Structure of Basic Implications of Output Controllability Types of Linear Time-Invariant Dynamical Systems

$$
\begin{array}{ccccc}
\overline{OC}(\mathrm{III}, 6666) & \Rightarrow & \overline{OC}(\mathrm{II}, 6666) & \Rightarrow & \overline{OC}(\mathrm{I}, 6666) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\overline{OC}(\mathrm{III}, 5555) & \Rightarrow & \overline{OC}(\mathrm{II}, 5555) & \Rightarrow & \overline{OC}(\mathrm{I}, 5555) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\overline{OC}(\mathrm{III}, 4444) & \Rightarrow & \overline{OC}(\mathrm{II}, 4444) & \Rightarrow & \overline{OC}(\mathrm{I}, 4444)
\end{array}
$$

Table 3a. Distinct Types of Observability of Linear Time-Invariant Dynamical Systems

Identified Variable \ Observed Variable	$y_j[t_o, t_1]$ $\exists j = 1, 2, \ldots, p$	$\underline{y}[t_o, t_1]$	$y_j[t_o, t_1]$ $\forall j = 1, 2, \ldots, p$
$\forall x_{io} \varepsilon \Sigma_i$ $\exists i = 1, 2, \ldots, n$	$\overline{OB}$(I, 4444) Selectively uniformly i-th state observable system	$\overline{OB}$(II, 4444) Proper i-th state observable system	$\overline{OB}$(III, 4444) Normal i-th state observable system
$\forall \underline{\tilde{x}}_o \varepsilon \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$	$\overline{OB}$(I, 5555) Selectively uniformly observable system in $\tilde{\Sigma}$	$\overline{OB}$(II, 5555) Proper observable system in $\tilde{\Sigma}$	$\overline{OB}$(III, 555) Normal observable system in $\tilde{\Sigma}$
$\forall \underline{x}_o \varepsilon \Sigma$	$\overline{OB}$(I, 6666) Selectively uniformly observable system	$\overline{OB}$(II, 6666) Proper observable system	$\overline{OB}$(III, 6666) Normal observable system

Table 3b. Structure of Basic Implications of Observability Types of Linear Time-Invariant Dynamical Systems

$$
\begin{array}{ccccc}
\overline{OB}(III, 6666) & \Rightarrow & \overline{OB}(II, 6666) & \Rightarrow & \overline{OB}(I, 6666) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\overline{OB}(III, 5555) & \Rightarrow & \overline{OB}(II, 5555) & \Rightarrow & \overline{OB}(I, 5555) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\overline{OB}(III, 4444) & \Rightarrow & \overline{OB}(II, 4444) & \Rightarrow & \overline{OB}(I, 4444)
\end{array}
$$

Table 4a. Distinct Types of State Invariance of Linear-Time Invariant Dynamical Systems

Invariant Quantity \ Input Variable	$u_j[t_o, t_1]$ $\exists_j = 1, 2, \ldots, r$	$\underline{u}_j[t_o, t_1]$	$u_j[t_o, t_1]$ $\forall_j = 1, 2, \ldots, k$
$x_i(t) \varepsilon \Sigma_i$, $i = 1,2,\ldots n$ $\forall x_{io} \varepsilon \Sigma_i$	$\overline{SI}$(I, 4444) Selectively uniformly i-th state invariant system	$\overline{SI}$(II, 4444) Proper i-th state invariant system	$\overline{SI}$(III, 4444) Normal i-th state invariant system
$\underline{\tilde{x}}(t) \varepsilon \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$ and $\underline{x}_o \varepsilon \tilde{\Sigma}$	$\overline{SI}$(I, 5555) Selectively uniformly state-invariant system in $\tilde{\Sigma}$	$\overline{SI}$(II, 5555) Proper state-invariant system in $\tilde{\Sigma}$	$\overline{SI}$(III, 5555) Normal state-invariant system in $\tilde{\Sigma}$
$\underline{x}(t) \varepsilon \Sigma$ $\forall \underline{x}_o \varepsilon \Sigma$	$\overline{SI}$(I, 6666) Selectively uniformly state-invariant system	$\overline{SI}$(II, 6666) Proper state-invariant system	$\overline{SI}$(III, 6666) Normal state-invariant system

Table 4b. Structure of Basic Implications of State Invariance Types of Linear Time-Invariant Dynamical Systems

$$
\begin{array}{ccccc}
\overline{SI}(\text{III}, 6666) & \Leftrightarrow & \overline{SI}(\text{II}, 6666) & \Rightarrow & \overline{SI}(\text{I}, 6666) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\overline{SI}(\text{III}, 5555) & \Leftrightarrow & \overline{SI}(\text{II}, 5555) & \Rightarrow & \overline{SI}(\text{I}, 5555) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\overline{SI}(\text{III}, 4444) & \Leftrightarrow & \overline{SI}(\text{II}, 4444) & \Rightarrow & \overline{SI}(\text{I}, 4444)
\end{array}
$$

Table 5a. Distinct Types of Output Invariance of Linear Time-Invariant Dynamical Systems

Invariant Quantity \ Input Variable	$u_j[t_o, t_1]$ j, $j = 1, 2, \ldots, r$	$\underline{u}[t_o, t_1]$	$u_j[t_o, t_1]$ j, $j = 1, 2, \ldots, r$
$y_i(t) \in R_i$, $i = 1, 2, \ldots, p$ $\forall y_{io} \in R_i$	$\overline{OI}$(I, 4444) Selectively uniformly i-th output invariant system	$\overline{OI}$(II, 4444) Proper i-th output invariant system	$\overline{OI}$(III, 4444) Normal i-th output invariant syst system
$\tilde{\underline{y}}(t) \in R^q$, $R^q \subset R^p$ $\forall \tilde{\underline{y}}_o \in R^q$	$\overline{OI}$(I, 5555) Selectively uniformly output in-variant system in R^q	$\overline{OI}$(II, 5555) Proper out-put invari-ant system in R^q	$\overline{OI}$(III, 5555) Normal output invariant system in R^q
$\underline{y}(t) \in R^p$ $\forall \underline{y}_o \in R^p$	$\overline{OI}$(I, 6666) Selectively uniformly output-invariant system	$\overline{OI}$(II, 6666) Proper out-put invari-ant system	$\overline{OI}$(III, 6666) Normal output-invariant system

Table 5b. Structure of Basic Implications of Output Invariance Types of Linear Time-Invariant Dynamical Systems

$$
\begin{array}{ccccc}
\overline{OI}(III,\ 6666) & \Longleftrightarrow & \overline{OI}(II,\ 6666) & \Longleftrightarrow & \overline{OI}(I,\ 6666) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\overline{OI}(III,\ 5555) & \Longleftrightarrow & \overline{OI}(II,\ 5555) & \Longleftrightarrow & \overline{OI}(I,\ 5555) \\
\Downarrow & & \Downarrow & & \Downarrow \\
\overline{OI}(III,\ 4444) & \Longleftrightarrow & \overline{OI}(II,\ 4444) & \Longleftrightarrow & \overline{OI}(I,\ 4444)
\end{array}
$$

the i-th row vector of the matrix A, and by

$$\underline{a}_{\cdot j} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}, \; j = 1, 2, \ldots, n$$

the j-th column vector of A. Hence

$$A = \begin{bmatrix} \underline{a}_{1\cdot} \\ \vdots \\ \underline{a}_{m\cdot} \end{bmatrix} = [\underline{a}_{\cdot 1} \cdots\cdots \underline{a}_{\cdot n}]$$

Consider now the linear time-invariant system (5.2). We shall denote by $\Gamma(s)$ the matrix

$$\Gamma(s) = [I s - A]$$

where s is some variable. Usually this will be the variable in the frequency domain of a Laplace transform of some time function, i.e., $\Gamma(s) = \mathcal{L}\{\Gamma(t)\}$. The determinant of a matrix will be denoted by det, f.e.

$$\det \Gamma(s) = \det [Is-A]$$

By

$$
{}^{i}\Gamma^{j}(s) = \begin{bmatrix}
s-a_{11} & -a_{12} & \cdots & -a_{1i-1} & b_{1j} & -a_{1,i+1} & \cdots & -a_{in} \\
-a_{21} & s-a_{22} & \cdots & -a_{2,i-1} & b_{2j} & -a_{2,i+1} & & -a_{2n} \\
-a_{31} & -a_{32} & \cdots & -a_{3i-1} & b_{3j} & -a_{3,i+1} & & -a_{3n} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
-a_{n1} & -a_{n2} & \cdots & -a_{ni-1} & b_{nj} & -a_{n,i+1} & & s-a_{nn}
\end{bmatrix}
$$

we shall denote the matrix $\Gamma(s)$ with i-th column substituted by the j-th column of some other matrix B.

Now we shall introduce the principal theorems.

Theorem $\overline{SC}$(I, 4444). The linear time-invariant system (5.2) is selectively uniformly i-th state controllable by the j-th input component $u_{j[t_o, t_1]}$, $j = 1, 2, \ldots, r$, with $u_{k[t_o, t_1]} = 0$ for $k \neq j$, i.e., the system is of the type $\overline{SC}$(I, 4444), if and only if det ${}^{i}\Gamma^{j}(s)$ is not equal to zero. Here, the matrix ${}^{i}\Gamma^{j}(s)$ is obtained from the matrix $\Gamma(s)$ replacing its i-th column by the j-th column of the matrix B.

Proof. Assume that the system is controllable, i.e., there exists some control $\underline{u}_j^*(t) = [0, \ldots, u_j^*(t), \ldots 0]^T$ *) such that the value of the i-th state component x_{io} at time $t_o = 0$ can be changed to any other value x_{i1} at time $t_1 > t_o$. Substituting $\underline{u}_j^*(t)$ into system equation (5.2), we get

*) A^T denotes the transpose of the matrix A.

$$\begin{aligned} \underline{x}(t) &= A\,\underline{x}(t) + B\,\underline{u}_j^*(t) \\ &= A\,\underline{x}(t) + \underline{b}_{\cdot j}\,u_j^*(t) \end{aligned} \tag{6.1}$$

Taking the Laplace transform of the equation (6.1), we can write

$$s\,\underline{x}(s) - \underline{x}(o) = A\,\underline{x}(s) + \underline{b}_{\cdot j}\,u_j^*(s)$$

Without loss of generality we can assume $\underline{x}(o) = \underline{0}$, so that

$$(sI - A)\,\underline{x}(s) = \underline{b}_{\cdot j}\,u_j^*(s)$$

where I is the unity matrix. In expanded form we have

$$\begin{array}{rcl} (s-a_{11})\,x_1(s) - a_{12}\,x_2(s) - \dots - a_{1n}x_n(s) &=& b_{1j}\,u_j^*(s) \\ -a_{21}\,x_1(s) + (s-a_{22})\,x_2(s) \dots - a_{2n}\,x_n(s) &=& b_{2j}\,u_j^*(s) \\ \text{-------------------------------} & & \\ -a_{n1}\,x_1(s) - a_{n2}\,x_2(s) - \dots + (s-a_{nn})\,x_n(s) &=& b_{nj}\,u_j^*(s) \end{array} \tag{6.2}$$

The solution of the above system of equations for $x_i(s)$, $i = 1, 2, \dots, n$, is by Cramer's rule unique for $\det \Gamma(s) \neq 0$ and is equal to

$$x_i(s) = \frac{\det \begin{bmatrix} s-a_{11} & -a_{12} & - \cdots & -a_{1,i-1} & b_{1j}\, u_j^*(s) & -a_{1,i+1} - & \cdots & -a_{1n} \\ -a_{21} & s-a_{22} & - \cdots & -a_{2,i-1} & b_{2j}\, u_j^*(s) & -a_{2,i+1} - & \cdots & -a_{2n} \\ \hdashline -a_{n1} & -a_{n2} & - \cdots & -a_{n,i-1} & b_{nj}\, u_j^*(s) & -a_{n\ i+1} - & \cdots & -s-a_{nn} \end{bmatrix}}{\det \Gamma(s)}$$

It is clear by inspection that the determinant of the matrix $\Gamma(s)$ for the system (5.2) is never identically zero for arbitrary values of s, hence the above solution is unique. The numerator of the above expression is the coefficient matrix of the left hand side of the system of Eq. (6.2) with the i-th column replaced by the expressions of the right side. We can rewrite it as follows:

$$x_i(s) = \frac{\det \begin{bmatrix} s-a_{11} & -a_{12} & - \cdots & -a_{1,i-1} & b_{1j} & -a_{1,i+1} & - \cdots & -a_{1n} \\ -a_{21} & s-a_{22} & - \cdots & -a_{2i-1} & b_{2j} & -a_{2,i+1} & - \cdots & -a_{2n} \\ \hdashline -a_{n1} & -a_{n2} & - \cdots & -a_{n,i-1} & b_{nj} & -a_{n,i+1} & - \cdots & s-a_{nn} \end{bmatrix} \cdot u_j^*(s)}{\det \Gamma(s)}$$

or

$$x_i(s) = \frac{\det {}^i\Gamma^j(s)}{\det \Gamma(s)} \cdot u_j^*(s)$$

Clearly, since the order of det ${}^i\Gamma^j(s)$ is always smaller than the order of det $\Gamma(s)$ = det $[sI-A]$, the necessary condition that the solution

$$x_i(t_1) = {}^{-1}\{x_i(s)\} = \int_o^{t_1} \Phi\ (t,\ \tau)\ u_j^*(t)\ dt$$

would exist for arbitrary values $x_i(t_1)$ for some $t_1 > t_o$ is that

$$\det {}^i\Gamma^j(s) \neq 0\ .$$

Sufficiency of this condition is self-explanatory.

Example 1. The $x_1(t)$ component of the system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}(t)$$

is controllable by the $u_1(t)$ component of the input vector $\underline{u}(t)$ because

$$\det {}^1\Gamma^1(s) = \det \begin{bmatrix} 1 & -1 \\ 0 & s \end{bmatrix} \neq 0$$

It is also controllable by the $u_2(t)$ component of the input

$\underline{u}(t)$ since

$$\det {}^{1}\Gamma^{2}(s) = \det \begin{bmatrix} 0 & -1 \\ 1 & s \end{bmatrix} \neq 0$$

The component $x_2(t)$ of the system is, however, not controllable by the component $u_1(t)$ of the input $\underline{u}(t)$ since

$$\det {}^{2}\Gamma^{1}(s) = \det \begin{bmatrix} s-1 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

An alternative formulation of the invariance criterion, which is often convenient to apply in design problems, is given by the following.

Corollary $\overline{\text{SC}}$ (I, 4444). The linear time-invariant system (5.2) is selectively uniformly i-th state controllable by the j-th input component, if and only if there is at least one non-zero element in the i-th row of the matrix

$$Q_j = \left[\underline{b}_{\cdot j} \vdots A\, \underline{b}_{\cdot j} \vdots \ldots \vdots A^{n-1}\, \underline{b}_{\cdot j}\right]$$

where $\underline{b}_{\cdot j}$ is the j-th column vector of the matrix B.

Proof. With $\underline{u}_j^*(t) = [0, \ldots, 0, u_j(t), 0 \ldots, 0]^T$, where $u_j(t)$ is the j-th component of the vector $\underline{u}(t)$, and $t_o = 0$, the solution of the differential equation (5.2) is

$$\underline{x}(t) = e^{At}\,\underline{x}_o + \int_0^t e^{A(t-\tau)}\,B\,\underline{u}_j^*(\tau)\,d\tau =$$

$$e^{At}\,\underline{x}_o + \int_0^t e^{A(t-\tau)}\,\underline{b}_{\cdot j}u_j(\tau)\,d\tau$$

The i-th component of the state vector $\underline{x}(t)$ is then equal to

$$x_i(t) = [e^{At}]_{i\cdot}\,\underline{x}_o + \int_0^t [e^{A(t-\tau)}]_{i\cdot}\,\underline{b}_{\cdot j}u_j(\tau)\,d\tau$$

where

$[e^{At}]_{i\cdot}$ and $[e^{A(t-\tau)}]_{i\cdot}$ are the i-th row vectors of the matrix e^{At} and $e^{A(t-\tau)}$, respectively.

Since, by hypothesis, the component $x_i(t)$ of the state vector $\underline{x}(t)$ is controllable, there exists time $t_1 > t_o$ and some input $u_j^*(t) \varepsilon\, U$ such that

$$0 = \left[e^{At_1}\right]_{i\cdot}\underline{x}_o + \int_0^{t_1}\left[e^{A(t-\tau)}\right]_{i\cdot}\,\underline{b}_{\cdot j}u_j^*(\tau)\,d\tau$$

or

$$\int_0^{t_1}\left[e^{A(t_1-\tau)}\right]_{i\cdot}\,\underline{b}_{\cdot j}u_j^*(t)\,d\tau = -\left[e^{At_1}\right]_{i\cdot}\underline{x}_o \qquad (6.3)$$

Observing that

$$e^{A(t_1 - \tau)} = e^{-A(\tau - t_1)} =$$

$$\sum_{p=0}^{\infty} (-1)^p A^p \frac{(\tau - t_1)^p}{p!}$$

and since, according to Cayley-Hamilton theorem, every n x n matrix satisfies its characteristic equation, it follows that

$$e^{-A(\tau - t_1)} = \sum_{\alpha=0}^{n-1} f_\alpha(\tau) A^\alpha$$

where the $f_\alpha(\tau)$ are scalar valued functions of τ. Substituting in Eq. (6.3) we obtain

$$\int_o^{t_1} \left[\sum_{\alpha=0}^{n-1} f_\alpha(\tau) A^\alpha\right]_{i\cdot} \underline{b}_{\cdot j} u_j^*(\tau) \, d\tau = - \left[e^{At_1}\right]_{i\cdot} \underline{x}_o$$

This is equal to

$$\sum_{\alpha=0}^{n-1} \left[A^\alpha\right]_{i\cdot} \underline{b}_{\cdot j} \, g_\alpha \, (t_1) = - \left[e^{At_1}\right]_{i\cdot} \underline{x}_o \qquad (6.4)$$

where

$$g_\alpha(t_1) = \int_o^{t_1} f_\alpha(\tau) u_j^*(\tau) \, d\tau$$

For the Eq. (6.4) to hold for arbitrary initial conditions $\underline{x}_o$, the necessary condition is that the coefficients of $g_\alpha(t_1)$ would not be identically zero, i.e., that at least one value

$$[A^\alpha]_{i.}\underline{b}_{.j} \neq 0$$

for $\alpha = 0, 1, \ldots, n-1$. This is equivalent to saying that the i-th row of the matrix

$$Q_j = [\underline{b}_{.j} \vdots A\, \underline{b}_{.j} \vdots \ldots \vdots A^{n-1}\, \underline{b}_{.j}]$$

would contain at least one non-zero element.

To prove the sufficiency, we note that in Eq. (6.3) we can always choose

$$u_j^*(\tau) = \frac{k}{t_1\ [e^{A(t_1 - \tau)}]_{i.}\underline{b}_{.j}}$$

where k is a real number equal to

$$k = -[e^{At_1}]_{i.}\underline{x}_o \qquad \text{Q.E.D.}$$

Example 2. In the preceding example of the system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}(t)$$

we have for $\underline{b}_{\cdot 1} = [1 \quad 0]^T$

$$Q_1 = [\underline{b}_{\cdot 1} \vdots Ab_{\cdot 1}] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

As we see, there are non-zero elements in the 1-st row of the matrix Q_1. Hence, the component $x_1(t)$ of the state vector $\underline{x}(t)$ is controllable by the component $u_1(t)$ of the input vector $\underline{u}(t)$. On the other hand, the second row of the matrix Q_1 has no non-zero elements, and therefore the component $x_2(t)$ is <u>not</u> controllable by the component $u_2(t)$.

These results are the same as the results obtained using previous criteria, which, of course, had to be expected.

The generalization to more than one component of the control input vector $\underline{u}(t)$ is straightforward, and will be presented here without proof.

<u>Theorem $\overline{SC}$(II, 4444)</u>: The i-th state component of the system (5.2) is proper if and only if

$$\det {}^i\Gamma^j(s) \neq 0$$

for some j, $j = 1, 2, \ldots, r$

<u>Example 3</u>. The component $x_2(t)$ of the system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \underline{u}(t)$$

is proper, since

$$\det {}^{2}\Gamma^{1}(s) = \det \begin{bmatrix} s-1 & 0 \\ -1 & 1 \end{bmatrix} \neq 0$$

even though

$$\det {}^{2}\Gamma^{2}(s) = \det \begin{bmatrix} s-1 & 0 \\ -1 & 0 \end{bmatrix} = 0$$

However, the component $x_1(t)$ of this system is not proper, since both

$$\det {}^{1}\Gamma^{1}(s) = \det \begin{bmatrix} 0 & 0 \\ 1 & s-2 \end{bmatrix} = 0$$

and

$$\det {}^{1}\Gamma^{2}(s) = \det \begin{bmatrix} 0 & 0 \\ 0 & s-2 \end{bmatrix} = 0$$

<u>Corollary $\overline{SC}$(II, 4444)</u>. The i-th state component of the system (5.2) is proper if and only if there is at least one non-zero element in the i-th row of the matrices

$$Q_j = [\underline{b}_{\cdot j} \vdots A\, \underline{b}_{\cdot j} \vdots \ldots \vdots A^{n-1}\, \underline{b}_{\cdot j}]$$

for some $j = 1, 2, \ldots, r$.

Example 4. In the previous example of the system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \underline{u}(t)$$

we have

$$Q_1 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

and

$$Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

Since no element in the first row of Q_1 and Q_2 is non-zero, the $x_1(t)$ component of $\underline{x}(t)$ is not proper. The component $x_2(t)$ is proper.

Theorem $\overline{\mathrm{SC}}$(III, 4444). The linear time-invariant system (5.2) is i-th state-normal if and only if

$$\det {}^{i}\Gamma^{j}(s) \neq 0$$

for all j, j = 1, 2, ..., r.

Corollary $\overline{\mathrm{SC}}$(III, 4444). The linear time-invariant system (5.2) is i-th state-normal if and only if the i-th rows of each of the matrices

$$Q_j = [\underline{b}_{\cdot j} \vdots A\,\underline{b}_{\cdot j} \vdots \ldots \vdots A^{n-1}\,\underline{b}_{\cdot j}]$$

have at least one non-zero element for all $j = 1, 2, \ldots, r$.

Example: For the system in the preceding example, i.e.,

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \underline{u}(t) ,$$

neither the component $x_1(t)$ nor the component $x_2(t)$ of the state vector is normal.

Next we shall give without proof the well known criteria for complete controllability, which for linear time invariant systems implies (and is implied by) state-proper system.

Theorem $\overline{\text{SC}}$(II, 6666). The system (5.2) is state-proper (or completely state controllable) if the composite matrix

$$Q = [B \vdots AB \vdots A^2B \vdots \ldots \vdots A^{n-1}\,B]$$

is of full rank [15].

Example 5. The system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \underline{u}(t)$$

is state-proper since the matrix

$$Q = [B \vdots AB] = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 2 & 4 & 4 \end{bmatrix}$$

is of full rank (i.e., of rank 2).

It is easy to see how Theorem $\overline{SC}$(II, 6666) can be applied to obtain the necessary and sufficient conditions for selectively uniformly state-controllable [$\overline{SC}$(I, 6666)] and state-normal [$\overline{SC}$(II, 6666)] systems. Observing that

$$B\ \underline{u}(t) = [\underline{b}_{\cdot 1} ; \underline{b}_{\cdot 2} \cdots \underline{b}_{\cdot r}] \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

$$= \underline{b}_{\cdot 1} u_1(t) + \underline{b}_{\cdot 2} u_2(t) + \ldots + \underline{b}_{\cdot r} u_r(t)$$

we can immediately formulate

Theorem $\overline{SC}$(I, 6666). The system (5.2) is selectively uniformly state-controllable by the input component $u_j(t)$, $j = 1, 2, \ldots r$, if and only if the n x n matrix

$$Q_j = [\underline{b}_{\cdot j} \vdots A\ \underline{b}_{\cdot j} \vdots A^2\ \underline{b}_{\cdot j} \vdots \ldots A^{n-1}\ \underline{b}_{\cdot j}]$$

is of rank n, or $\det Q_j \neq 0$.

Theorem $\overline{SC}$(III, 6666). The system (5.2) is state-normal if and only if the n x n matrices

$$Q_j = [\underline{b}_{\cdot j} \vdots A\,\underline{b}_{\cdot j} \vdots A^2\,\underline{b}_{\cdot j} \vdots \ldots \vdots A^{n-1}\,\underline{b}_{\cdot j}]$$

are of rank n for all $j = 1, 2, \ldots, r$, or $\det Q_j \neq 0$ for all $j = 1, 2, \ldots, r$.

Example 6. The system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \underline{u}(t)$$

is selectively uniformly state controllable because

$$\det Q_2 = \det \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \neq 0$$

but it is not state normal since

$$\det Q_1 = \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$$

Weiss and Kalman have shown that the set $R(Q)$ of all controllable states of the system (5.2) span a subspace of Σ which is invariant under the transformation A. [286]. The basis of this set of controllable states is the set of linearly independent column vectors of the matrix Q (or Q_j for some $j = 1, 2, \ldots, r$) [15]. We shall now state and prove theorems regarding the controllability of the system (5.2) in some subspace of the state space Σ.

Theorem $\overline{SC}(I, 5555)$. The system (5.2) is selectively uniformly state controllable in the subspace $\tilde{\Sigma}$ of the state space Σ, if and only if the projection of the set of controllable states, Q_c, on the subspace $\tilde{\Sigma}$ spans the whole subspace $\tilde{\Sigma}$.

Proof. Assume the system (5.2) is selectively uniformly state controllable in the subspace $\tilde{\Sigma}$ of dimension $k < n$. Then there exists a set of vectors $\{\underline{\tilde{x}}_1, \underline{\tilde{x}}_2, \ldots, \underline{\tilde{x}}_k\}$ in $\tilde{\Sigma}$ which span the whole of $\tilde{\Sigma}$ i.e., are the basis of $\tilde{\Sigma}$. Let E_k be the projection operator which projects controllable states $\underline{x} \in Q_c$ on $\tilde{\Sigma}$. Obviously, Q_c must contain at least k column vectors $\underline{x}_1, \ldots, \underline{x}_k$ such that $E_k \underline{x}_1 = \underline{\tilde{x}}_1$, $E_k \underline{x}_2 = \underline{\tilde{x}}_2, \ldots, E_k \underline{x}_k = \underline{\tilde{x}}_k$. Hence, the projection of Q_c on the subspace $\tilde{\Sigma}$ spans the whole subspace.

To prove the "only if" portion of the theorem, assume that the projection of Q_c on $\tilde{\Sigma}$ spans the whole of $\tilde{\Sigma}$. Then there is a set of controllable states $\{\underline{\tilde{x}}_1, \underline{\tilde{x}}_2, \ldots, \underline{\tilde{x}}_k\}$ in $\tilde{\Sigma}$ such that any other element in $\tilde{\Sigma}$ is a linear combination of these vectors. Hence, the system is controllable in $\tilde{\Sigma}$.

Corollary $\overline{SC}(I, 5555)$. Consider the system (5.2) and let Q_j be the $n \times n$ matrix defined

$$Q_j = [\underline{b}_{\cdot j} \vdots \ldots. \vdots A^{n-1} \underline{b}_{\cdot j}]$$

Further, let E_k be the projector (linear transformation operator) such that $E_k\underline{x} = \underline{\tilde{x}}$ and $\underline{\tilde{x}} \in \tilde{\Sigma}$ for all $\underline{x} \in \Sigma$. Then the system is selectively uniformly state controllable in the subspace $\tilde{\Sigma}$ of dimension k by the input component $u_j(t)$ if and only if the matrix

$$E_k Q_j = [E_k \underline{b}_{\cdot j} \vdots E_k A \underline{b}_{\cdot j} \vdots \ldots \vdots E_k A^{n-1} \underline{b}_{\cdot j}]$$

is of the rank $r \geq k$, where k is the dimension of the subspace $\tilde{\Sigma}$.

The proof follows immediately from the preceding theorem.

Example 7. Consider the linear time-invariant dynamical system given by the equation

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{u}(t)$$

Is that system selectively uniformly controllable by $u_1(t)$ in the subspace $\tilde{\Sigma}$ generated by the vectors $[x_1, 0, 0]^T$ and $[0, x_2, 0]^T$? Well, the set of controllable states by $u_1(t)$ is the linearly independent set of the column vectors of the matrix

$$Q_1 = [\underline{b}_{\cdot 1} \vdots A \underline{b}_{\cdot 1} \vdots A^2 \underline{b}_{\cdot 1}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

The projector of the set $\{\underline{q}_{\cdot 1}\}$ on $\tilde{\Sigma}$ is the matrix

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the projection of the generators set $\{\underline{q}_{\cdot 1}\}$ on the subspace $\tilde{\Sigma}$,

$$E_2\, Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

contains only one linear independent vector, i.e., the matrix $E_2\, Q_1$ is of rank $r = 1$ with $r < k$ $(k = 2)$, the system is not controllable in the subspace $\tilde{\Sigma}$ generated by the vectors $[x_1, 0, 0]^T$ and $[0, x_2, 0]^T$. In other words, it is not controllable in the (x_1, x_2) plane.

Is it controllable in the subspace $\tilde{\Sigma}$ generated by the vectors $[x_1, 0, 0]^T$ and $[0, 0, x_3]^T$, viz. in the (x_1, x_3) plane? With matrix Q_1 as above and projector

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we obtain

$$E_2\, Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

The above product matrix is of rank $r = 2 = k$. Hence, the system is controllable in the subspace $\tilde{\Sigma}$, representing the (x_1, x_3) plane.

Similarly, we can verify that the system is also controllable in the subspace $\tilde{\Sigma}$ generated by the vectors $[0, x_2, 0]^T$ and $[0, 0, x_3]^T$.

We can now easily formulate the necessary and sufficient conditions for the remaining two types of state controllability, the proofs of which follow immediately from the preceding theorems.

<u>Theorem $\overline{SC}$(II, 5555)</u>. The system (5.2) is state proper in the subspace $\tilde{\Sigma}$ of the state space Σ if and only if the $n \times nr$ matrix

$$E_k\, Q \;=\; [E_k\, B \;\vdots\; E_k\, A\, B \;\vdots\; E_k\, A^2\, B \;\vdots\; \ldots\; E_k\, A^{n-1}\, B]$$

is of the rank $r \geqslant k$, where k is the dimension of the subspace $\tilde{\Sigma}$ and E_k is the projector of the set of controllable states Q_c on $\tilde{\Sigma}$ along the space $\tilde{\Sigma}^{\perp}$, the orthogonol complement of the subspace $\tilde{\Sigma}$.

<u>Example 8</u>. Consider the system of previous example. Is this system state proper (or completely state-controllable) in the subspace $\tilde{\Sigma}$ generated by the vectors $[x_1, 0, 0]^T$ and $[0, x_2, 0]^T$?

Now, we have

$$E_2\, Q \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{bmatrix} \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which matrix is of rank $r = 1$, $r < k$. Hence, the system is not proper in the subspace $\tilde{\Sigma}$, i.e., the (x_1, x_2) - plane.

It is, however, proper in the subspaces of Σ representing the (x_1, x_3) - plane and (x_2, x_3) - plane.

<u>Theorem $\overline{SC}$(III, 5555)</u>. The system (5.2) is state normal in the subspace $\tilde{\Sigma}$ of the state space Σ if and only if the $n \times n$ matrices

$$E_k\, Q_j \;=\; [E_k\, \underline{b}_{\cdot j} \;\vdots\; E_k\, A\, \underline{b}_{\cdot j} \;\vdots\; \ldots \;\vdots\; E_k\, A^{n-1}\, \underline{b}_{\cdot j}]$$

are of the rank $r \geqslant k$, where k is the dimension of the subspace $\tilde{\Sigma}$ and E_k is the projector of Q_c on $\tilde{\Sigma}$, for all $j = 1, 2, \ldots, r$.

Example 9: Consider the system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}(t)$$

Is this a state-normal system in the subspace $\tilde{\Sigma}$ generated by the vectors $[x_1, 0, 0]^T$ and $[0, x_2, 0]^T$?

Using the criterion of Theorem $\overline{SC}$(III, 5555), we verify that

$$E_2 Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

is of rank $r = 2$ and also

$$E_2 Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is of rank $r = 2$. Since for all j, $j = 1, 2$, the rank of $E_2 Q_j$ is equal to the dimension of the subspace $k = 2$, we conclude that the above system is state-normal in the (x_1, x_2) - plane.

6.2. Necessary and Sufficient Conditions for Output Controllability. The theorems of output-controllability are obtainable as an extension of those of state-controllability in a rather straightforward manner, although, as it has been shown by Kreindler and Sarachik [142] state-controllability is neither necessary nor sufficient condition for output controllability.

In the system under consideration, i.e.,

$$\underline{\dot{x}}(t) = A\,\underline{x}(t) + B\underline{u}(t)$$

$$\underline{y}(t) = F\,\underline{x}(t) + G\underline{u}(t)$$

the term $G\underline{u}(t)$ represents the so-called direct transmission from input to output. It is obvious that the existence of direct transmission has no effect on state-controllability, but it always aids output-controllability.

Theorem $\overline{OC}$(I, 4444). The linear time-invariant system (5.2) is selectively uniformly i-th output-controllable by the j-th input component $u_{j[t_o,\ t_1]}$, $j = 1, 2, \ldots, r$, if and only if

$$f_{i1} \det {}^{1}\Gamma^{j}(s) + f_{i2} \det {}^{2}\Gamma^{j}(s) + \ldots + f_{in} \det {}^{n}\Gamma^{j}(s) + g_{ij} \det\Gamma(s) \neq 0$$

f_{ij} is the i-th row j-th column element of the matrix F, g_{ij} is the i-th row j-th column element of the matrix G, and ${}^{i}\Gamma^{j}(s)$ are matrices $\Gamma(s)$ with the column vector $\underline{b}_{\cdot j}$ of the matrix B substituted for the i-th column of the matrix $\Gamma(s)$.

Proof. Taking the Laplace transforms of the Eg. (5.2), we can write

$$\underline{y}(s) = F\,\underline{x}(s) + G\,\underline{u}(s)$$

Assuming that system is i-th output j-th input controllable, there exists some control $\underline{u}_j^*(t) = [0, \ldots, 0, u_j^*(t), 0, \ldots, 0]^T$ such that the value of the i-th output component y_{io} at the time $t_o = 0$ can be changed to any other value at time $t_1 > t_o$. Substituting $\underline{u}_j^*(s)$ [i.e., Laplace transform of $\underline{u}_j^*(t)$], and taking only the i-th component of the vector $\underline{y}(s)$, we get

$$y_i(s) = \underline{f}_{i.}\, \underline{x}(s) + g_{ij}\, u_j^*(s) \tag{6.5}$$

where $\underline{f}_{i.}$ is the i-th row vector of the matrix F.

Assuming zero initial conditions for the state variable $\underline{x}(t)$, i.e., $\underline{x}(t_o) = \underline{x}_o = \underline{0}$, we can write, as it was shown in the proof of Theorem $\overline{SC}$(I, 4444) that

$$x_i(s) = \frac{\det {}^i\Gamma^j(s)}{\det \Gamma(s)}\, u_j^*(s)$$

Substituting this result into Eq. (6.5), we obtain

$$\begin{aligned}
y_i(s) &= \frac{1}{\det \Gamma(s)} \Big[f_{i1} \det {}^1\Gamma^j(s) + f_{i2} \det {}^2\Gamma^j(s) + \\
&\qquad \ldots + f_{in} \det {}^n\Gamma^j(s) \Big]\, u_j^*(s) + g_{ij} u_j^*(s) \\
&= \frac{1}{\det\Gamma(s)} \Big[f_{i1} \det {}^1\Gamma^j(s) + f_{i2} \det {}^2\Gamma^j(s) + \\
&\qquad \ldots + f_{in} \det {}^n\Gamma^j(s) + g_{ij} \det \Gamma(s) \Big]\, u_j^*(s)
\end{aligned}$$

Since, in general, $\det \Gamma(s) \neq 0$, the necessary condition that the inverse transforms $\mathcal{L}^{-1}\{y_i(s)\} = y_i(t)$ assumes any real value at time $t_1 > t_o$, is that

$$f_{i1} \det {}^1\Gamma^j(s) + f_{i2} \det {}^2\Gamma^j(s) + \\ \ldots + f_{in} \det {}^n\Gamma^j(s) + g_{ij} \det \Gamma(s) \neq 0$$

It follows immediately from what was said in the proof of Theorem $\overline{SC}(I, 4444)$ that this condition is also sufficient.

Corollary $\overline{OC}(I, 4444)$. The system (5.2) is selectively uniformly i-th output-controllable by the j-th input component if and only if there is at least one non-zero element in the i-th row of the $n \times (n + 1)$ matrix

$$P_j = [F\,\underline{b}_{\cdot j} \vdots F\,A\,\underline{b}_{\cdot j} \vdots F\,A^2\,\underline{b}_{\cdot j} \vdots \ldots \vdots F\,A^{n-1}\,\underline{b}_{\cdot j} \vdots \underline{g}_{\cdot j}]$$

The proof, which will be omitted here, follows a similar pattern like the proof of the Corollary $\overline{SC}(I, 4444)$. Furthermore, the above corollary can be considered a special case of the output controllability in some output subspace $R^q \subset R^p$, with the above condition derived from the conditions for the more general case discussed in the following paragraphs.

Example 10: Consider the system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \underline{x}(t) \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}(t)$$

Is this system selectively uniformly controllable in the $y_1(t)$ output component by the $u_1(t)$ input component? We form the matrix

$$P_1 = [F\,\underline{b}_{.1} \vdots F\,A\,\underline{b}_{.1} \vdots \underline{g}_{.1}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and observe that the first row of the matrix P_1 contains elements not all of which are zero. Hence, the output $y_1(t)$ is controllable by $u_1(t)$. However, the output $y_2(t)$ is not controllable by the input component $u_1(t)$, because there are no non-zero elements in the second row of the matrix.

The generalization of the necessary and sufficient conditions to proper and normal i-th output controllable systems is straightforward, and so we have

Theorem $\overline{OC}$(II, 4444). The system (5.2) is i-th output-proper (which implies complete controllability) if and only if

$$f_{i1} \det {}^{1}\Gamma^{j}(s) + f_{i2} \det {}^{2}\Gamma^{j}(s) + \dots$$

$$+ f_{in} \det {}^{n}\Gamma^{j}(s) + g_{ij} \det \Gamma(s) \neq 0$$

for some $j = 1, 2, \ldots, r$.

Or, alternatively, if and only if there is at least one non-zero element in the i-th row of the matrix

$$P_j = [F\,\underline{b}_{\cdot j} \vdots F\,A\,\underline{b}_{\cdot j} \vdots \ldots F\,A^{n-1}\,\underline{b}_{\cdot j} \vdots g_{\cdot j}]$$

for some $j = 1, 2, \ldots, r$.

Theorem $\overline{\mathrm{OC}}$(III, 4444). The system (5.2) is i-th output normal if and only if

$$f_{i1} \det {}^{1}\Gamma^{j}(s) + f_{i2} \det {}^{2}\Gamma^{j}(s) + \ldots + f_{in} \det {}^{n}\Gamma^{j}(s) + g_{ij} \det \Gamma(s) \neq 0$$

for all $j = 1, 2, \ldots, r$, or, alternatively, if and only if

$$P_j = [F\,\underline{b}_{\cdot j} \vdots F\,A\,\underline{b}_{\cdot j} \vdots \ldots \vdots F\,A^{n-1}\,\underline{b}_{\cdot j} \vdots g_{\cdot j}]$$

has at least one non-zero element in the i-th row for all $j = 1, 2, \ldots, r$.

Example 11. For the system of Example 10, we have

$$P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Hence, the system is both $y_1(t)$ and $y_2(t)$ output proper, and it is also $y_1(t)$ output normal, but it is not $y_2(t)$ output normal.

The necessary and sufficient conditions for output-proper systems [142] are contained in the following

Theorem $\overline{\mathrm{OC}}$(II, 6666). The system (5.2) is output proper (i.e., completely output controllable) if and only if the $p \times (n + 1)r$ composite matrix

$$P = [F\ B \ \vdots\ F\ A\ B \ \vdots\ F\ A^2\ B \ \vdots\ \ldots\ F\ A^{n-1}\ B \ \vdots\ D]$$

is of full rank (= p).

Example 12. For the system of Example 10, we have the matrix

$$P = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

the rank of which is 2. Hence the system is output proper.

Theorem $\overline{\mathrm{OC}}$(I, 6666). The system (5.2) is selectively uniformly output controllable by the $u_j(t)$ input component if and only if the $p \times (n + 1)$ matrix

$$P_j = [F\ \underline{b}_{\cdot j} \ \vdots\ F\ A\ \underline{b}_{\cdot j} \ \vdots\ \ldots\ \vdots\ F\ A^{n-1}\ \underline{b}_{\cdot j} \ \vdots\ \underline{g}_{\cdot j}]$$

is of full rank (i.e., of rank p).

Theorem $\overline{OC}$(III, 6666). The system (5.2) is output normal if and only if the $p \times (n + 1)$ matrices

$$P_j = [F\,\underline{b}_{\cdot j} \,\vdots\, F\,A\,\underline{b}_{\cdot j} \,\vdots\, \dots \,\vdots\, F\,A^{n-1}\,\underline{b}_{\cdot j} \,\vdots\, \underline{g}_{\cdot j}]$$

are of full rank for all $j = 1, 2, \dots, r$.

Example 13. For the system of Example 10, we had

$$P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Since both the matrix P_1 and P_2 are of rank $1 < p = 2$ (i.e., the matrices are not of full rank), therefore the system is neither selectively uniformly output controllable by $u_1(t)$ nor selectively uniformly output controllable by $u_2(t)$. Naturally, under these circumstances it cannot be output normal either.

We shall now state and prove a theorem about the set of controllable outputs.

Theorem 6.2.1. The set $R(P)$ of controllable outputs of the system (5.2) is a vector subspace of the output space R^p.

Proof. By definition, a linear time-invariant system is completely output controllable (i.e., output proper) if there exists a control input $\underline{u}_{[t_o,\ t_1]}$ such that any final output $\underline{y}(t_1)$ can be obtained starting with arbitrary initial conditions $\underline{y}(t_o)$, in finite period of time $[t_o,\ t_1]$. Let $\underline{y}_1(t_o)$ and $\underline{y}_2(t_o)$ be two such controllable points in the output space R^p. Without loss of generality, we can assume that the final output to be achieved is equal to $\underline{0}$. Assuming that the system is output-controllable, there exist inputs $\underline{u}_1^*(t)$ and $\underline{u}_2^*(t)$ which make this transition from the initial outputs $\underline{y}_1(t_o)$ and $\underline{y}_2(t_o)$ to $\underline{0}$ possible during the period of time $[t_o,\ t_1]$. Then we have

$$\underline{y}(t_1) = \underline{0} = F\, e^{A(t_1-t_o)}\, \underline{x}_{10} + \int_{t_o}^{t_1} F\, e^{A(t_1-\tau)}\, B\underline{u}_1^*(\tau)\, d\tau + G\underline{u}_1^*(t_1)$$

and

$$\underline{y}(t_1) = \underline{0} = F\, e^{A(t_1-t_o)}\, \underline{x}_{20} + \int_{t_o}^{t_1} F\, e^{A(t_1-\tau)}\, B\underline{u}_2^*(\tau)\, d\tau + Gu_2^*(t_1)$$

where the initial outputs are

$$\underline{y}_1(t_o) = F\, \underline{x}_{10} + G\underline{u}_1^*(t_1) \tag{6.6}$$

and

$$\underline{y}_2(t_o) = F\, \underline{x}_{20} + G\underline{u}_2^*(t_o) \tag{6.7}$$

To show that the set of controllable outputs $R(P)$ of the system (5.2) is a subspace of the output (vector) space R^P, we have to show that for any $\underline{y}_1(t_o) \in R(P)$ and $\underline{y}_2(t_o) \in R(P)$, the linear combination

$$\lambda \underline{y}_1(t_o) + \mu \underline{y}_2(t_o) \in R(P) \tag{6.8}$$

for arbitrary real values of λ and μ, is also a controllable output. This means, that one has to show that there exists some input $\underline{u}^*(t)$ over the period $[t_o, t_1]$ which affects the transition from the initial output

$$\underline{y}(t_o) = \lambda F \underline{x}_{10} + \lambda G \underline{u}_1^*(t_o) + \mu F \underline{x}_{20} + \mu G \underline{u}^*(t_o)$$

to the final output $\underline{y}(t_1) = \underline{0}$. Such a control input does exist because it is sufficient to choose $\underline{u}^*(t) = \underline{u}_1^*(t) + \underline{u}_2^*(t)$ to obtain

$$\underline{y}(t) = F e^{A(t-t_o)} (\lambda \underline{x}_{10} + \mu \underline{x}_{20}) + \int_{t_o}^{t} F e^{A(t-\tau)} B \underline{u}^*(t) d\tau + G \underline{u}^*(t)$$

$$= \lambda \left[F e^{A(t-t_o)} \underline{x}_{10} + \int_{t_o}^{\tau} F e^{A(t-\tau)} B \underline{u}_1^*(\tau) d\tau + G \underline{u}_1^*(t) \right]$$

$$+ \mu \left[F e^{A(t-t_o)} \underline{x}_{20} + \int_{t_o}^{\tau} F e^{A(t-\tau)} B \underline{u}_2^*(\tau) d\tau + G \underline{u}_2^*(t) \right]$$

which expression becomes equal to $\underline{0}$ for $t = t_1$.

Based on this theorem, we can now state the controllability conditions in some subspace of the output space, proof of which follows essentially the lines of the proof of Theorem $\overline{SC}(I, 5555)$ and will not be repeated here.

Theorem $\overline{OC}(I, 5555)$. The system (5.2) is selectively uniformly output controllable in the subspace R^q of the output space R^p, $q < p$, if and only if the projection of the set of controllable outputs, R(P), on the subspace R^q spans the whole subspace R^q.

Corollary $\overline{OC}(I, 5555)$. Consider the system (5.2) and let P_j be the $p \times (n + 1)$ composite matrix

$$P_j = [F\, \underline{b}_{.j} \vdots F\, A\, \underline{b}_{.j} \vdots F\, A^2\, \underline{b}_{.j} \vdots \ldots \vdots F\, A^{n-1}\, \underline{b}_{.j} \vdots \underline{g}_{.j}]$$

Further, let E_k be the projector such that for all $\underline{y} \in R^p$ and $\underline{\tilde{y}} \in R^q$, $E_k\, \underline{y} = \underline{\tilde{y}}$, where $R^q \subset R^p$. Then the system is selectively uniformly output controllable in the subspace R^q by the input component $u_j(t)$ if and only if the matrix

$$E_k\, P_j = [E_k\, F\, \underline{b}_{.j} \vdots E_k\, F\, A\, \underline{b}_{.j} \vdots \ldots . \vdots E_k\, F\, A^{n-1}\, \underline{b}_{.j} \vdots E_k\, \underline{g}_{.j}]$$

is of the rank $r \geq q$, where q is the dimension of the subspace R^q.

Example 14. Consider the system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 4 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \underline{u}(t)$$

Is this system selectively uniformly output controllable in the output subspace R^q generated by the vectors $[y_1, 0, 0]^T$ and $[0, y_2, 0]^T$? For the control input component $u_1(t)$ we obtain the matrix

$$P_1 = [F\, \underline{b}_{\cdot j} \vdots F\, A\, \underline{b}_{\cdot 1} \vdots F\, A^2\, \underline{b}_{\cdot 1} \vdots \underline{g}_{\cdot 1}] = \begin{bmatrix} 4 & 8 & 16 & 4 \\ 1 & 2 & 4 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix}$$

With the projector

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we obtain

$$E_2\, P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 4 & 8 & 16 & 4 \\ 1 & 2 & 4 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 16 & 4 \\ 1 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $p_2 p_1$ has the rank $r = 1$ which is less than the dimension of tne subspace R^q . Hence, the system is not selectively uniformly output controllable by the input $u_1(t)$ in the subspace R^q , which is (y_1, y_2) - plane. It is, however, controllable in the (y_1, y_3) plane and (y_2, y_3) plane, which can be easily verified by choosing appropriate projectors and calculating the ranks of resulting matrices.

With regard to the input component $u_2(t)$, the system is selectively uniformly output controllable in the output subspace represented by the (y_1, y_2) - plane and (y_1, y_3) - plane, but not in the (y_2, y_3) - plane.

Theorem $\overline{\text{OC}}$(II, 5555). The system (5.2) is output proper in the subspace R^q of the output space R^p if and only if the $p \times (n + 1)\, k$ matrix

$$E_k P = [E_k F B \vdots E_k F A B \vdots \dots \vdots E_k F A^{n-1} B \vdots E_k D]$$

has the rank $r \geqslant q$, where q is the dimension of the subspace R^q of the output space R^p, and E_k is the projector of the set of controllable outputs $R(P)$ on R^q along the space $R^{q\perp}$.

Theorem $\overline{\text{OC}}$(III, 5555). The system (5.2) is output normal in the subspace R^q of the output space R^p if and only if the $p \times (n + 1)$ matrices

$$E_k P_j = [E_k F \underline{b}_{\cdot j} \vdots E_k F A \underline{b}_{\cdot j} \vdots \dots \vdots E_k F A^{n-1} \underline{b}_{\cdot j} \vdots E_k \underline{g}_{\cdot j}]$$

have the rank $r \geqslant q$ for all $j = 1, 2, \dots, r$, where q is the dimension of the output subspace R^q and E_k is the appropriate projector

Example 15. In the preceding example, we have

$$E_k\, P = E_k \times \begin{bmatrix} 4 & 4 & 8 & 8 & 16 & 16 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{bmatrix}$$

and we can easily verify that the system is not output proper in either one of the two dimensional subspace R^2 of the output space R^3 generated by any two of the vectors $[y_1, 0, 0]^T$, $[0, y_2, 0]^T$ and $[0, 0_1, y_3]^T$. Nor is the system output normal.

6.3. Necessary and Sufficient Conditions for Observability. Necessary and sufficient conditions for observability can be directly related to those of state controllability by making use of Kalman's duality principle [114]. The duality principle is contained in the following

Theorem (6.1). The linear time-invariant system

$$\begin{aligned} \dot{\underline{x}}(t) &= A\,\underline{x}(t) + B\,\underline{u}(t) \\ \underline{y}(t) &= F\,\underline{x}(t) + G\,\underline{u}(t) \end{aligned} \tag{5.2}$$

is observable (in the sense of any of the definitions given in the Table 3 of the Appendix) if and only if the adjoint system

$$\begin{aligned} \hat{\underline{x}}(t) &= -A^T\,\hat{\underline{x}}(t) + F^T\,\hat{\underline{u}}(t) \\ \hat{\underline{y}}(t) &= B^T\,\hat{\underline{x}}(t) + G\,\hat{\underline{u}}(t) \end{aligned} \tag{6.9}$$

is controllable (in the corresponding sense of any of the definitions given in the Table 1 of the Appendix).

It is also known that the set of observable states R(H) is a subspace of Σ [15] and the orthogonal component of it, i.e., the set of unobservable states $R(H)^{\perp}$, is an invariant subspace under transformation A.

Applying the above results, we obtain the following theorems regarding the necessary and sufficient criteria for different types of observability.

Theorem $\overline{OB}$(I, 4444). The system (5.2) is selectively uniformly i-th state observable with respect to $y_j(t)$ output component if and only if $\det {}^{i}\tilde{\Gamma}^{j}(s) \neq 0$ where $\tilde{\Gamma}(s)$ is the matrix

$$\tilde{\Gamma}(s) = [Is + A^T]$$

and the matrix ${}^{i}\tilde{\Gamma}^{j}(s)$ is obtained from the matrix $\tilde{\Gamma}(s)$ by replacing its i-th column by the j-th column of the matrix F^T, or, alternatively, if and only if there is at least one non-zero element in the i-th row of the matrix

$$H_j = [\underline{f}^T_{\cdot j} \vdots A^T \underline{f}^T_{\cdot j} \vdots (A^T)^2 \underline{f}^T_{\cdot j} \vdots \ldots \vdots (A^T)^{n-1} \underline{f}^T_{\cdot j}]$$

where $\underline{f}^T_{\cdot j}$ is the j-th column (vector) of the matrix F^T.

Theorem $\overline{OB}$(II, 4444). The system (5.2) is proper i-th state observable if and only if

$$\det {}^{i}\tilde{\Gamma}^{j}(s) \neq 0$$

for some $j = 1, 2, \ldots, p$, or, alternatively, if and only if not all the elements in the i-th rows of the n x n matrices

$$H_j = [\underline{f}^T_{.j} \vdots A^T \underline{f}^T_{.j} \vdots \ldots \vdots (A^T)^{n-1} \underline{f}^T_{.j}]$$

are zeroes for some $j = 1, 2, \ldots, p$.

<u>Theorem $\overline{OB}$(III, 4444)</u>. The system (5.2) is normal i-th state observable if and only if

$$\det {}^{i}\tilde{\Gamma}^{j}(s) \neq 0$$

for all $j = 1, 2, \ldots, p$, or, alternatively, if and only if the i-th row or each of the matrices

$$H_j = [\underline{f}^T_{.j} \vdots A^T \underline{f}^T_{.j} \vdots \ldots \vdots (A^T)^{n-1} \underline{f}^T_{.j}]$$

has at least one non-zero element for all $j = 1, 2, \ldots, p$.

<u>Theorem $\overline{OB}$(I, 5555)</u>. The system (5.2) is selectively uniformly observable in the k-dimensional subspace $\tilde{\Sigma}$ of the state space Σ , if and only if the projection of the set of observable states R(H) on the subspace of $\tilde{\Sigma}$ spans the whole subspace of $\tilde{\Sigma}$, or alternatively, if and only if the matrix

$$E_k H_j = [E_k \underline{f}^T_{.j} \vdots E_k A^T \underline{f}^T_{.j} \vdots \ldots \vdots E_k (A^T)^{n-1} \underline{f}^T_{.j}]$$

has the rank $r \geqslant k$, where E_k is the projector of R(H) on $\tilde{\Sigma}$ and k is the dimension of the subspace $\tilde{\Sigma}$.

<u>Theorem $\overline{OB}$(II, 5555)</u>. The system (5.2) is a proper observable system in the subspace $\tilde{\Sigma}$ of the state space Σ if and only if the $n \times np$ matrix

$$E_k H = [E_k F^T \vdots E_k A^T F^T \vdots E_k (A^T)^2 F^T \vdots \ldots \vdots E_k (A^T)^{n-1} F^T]$$

is of the rank $r \geqslant k$, where k and E_k are defined as above.

<u>Theorem $\overline{OB}$(III, 5555)</u>. The system (5.2) is a normal observable system in the subspace $\tilde{\Sigma}$ of the state space Σ if and only if the $n \times n$ matrices

$$E_k H_j = [E_k \underline{f}^T_{\cdot j} \vdots E_k A^T \underline{f}^T_{\cdot j} \vdots \ldots \vdots E_k (A^T)^{n-1} \underline{f}^T_{\cdot j}]$$

are of the rank $r \geqslant k$ for all $j = 1, 2, \ldots, p$, where k and E_k are defined as above.

<u>Theorem $\overline{OB}$(I, 6666)</u>. The system (5.2) is selectively uniformly observable with respect to the j-th output component $y_j(t)$, $j = 1, 2, \ldots, p$, if and only if the $n \times n$ matrix

$$H_j = [\underline{f}^T_{\cdot j} \vdots A^T \underline{f}^T_{\cdot j} \vdots (A^T)^2 \underline{f}^T_{\cdot j} \vdots \ldots \vdots (A^T)^{n-1} \underline{f}^T_{\cdot j}]$$

is of full rank, i.e., rank n, so that $\det H_j \neq 0$.

<u>Theorem $\overline{OB}$(II, 6666)</u>. The system is (5.2) is a proper observable one (or completely observable) if and only if the composite $n \times nk$ matrix

$$H = [F^T \vdots A^T F^T \vdots (A^T)^2 F^T \vdots \ldots \vdots (A^T)^{n-1} F^T]$$

is of full rank, i.e. rank n .

Theorem $\overline{OB}$(III, 6666). The system (5.2) is a normal observable one if and only if the $n \times n$ matrices

$$H_j = [\underline{f}^T_{\cdot j} \vdots A^T \underline{f}^T_{\cdot j} \vdots (A^T)^2 \underline{f}^T_{\cdot j} \vdots \ldots \vdots (A^T)^{n-1} \underline{f}^T_{\cdot j}]$$

are of rank n, so that $\det H_j \neq 0$, for all $j = 1, 2, \ldots, p$.

Example 16. Consider the system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \underline{u}(t)$$

for which

$$H_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} ; \quad H_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 1 \end{bmatrix}$$

Hence, by theorem

$\overline{OB}$(I, 4444) - the system is selectively uniformly $x_2(t)$ and $x_3(t)$ observable with respect to output $y_1(t)$, and $x_1(t)$, $x_2(t)$, and $x_3(t)$ observable with respect to output $y_2(t)$.

$\overline{OB}$(II, 4444) - the system is $x_1(t)$, $x_2(t)$, and $x_3(t)$ observable proper.

$\overline{OB}$(III, 4444) - the system is not normal $x_1(t)$ observable, but it is so with respect to the state components $x_2(t)$ and $x_3(t)$.

$\overline{OB}$(I, 5555) - the system is selectively uniformly observable by the $y_1(t)$ output in the (x_2, x_3) plane, but it is not observable in the (x_1, x_2) and (x_1, x_3) - planes.

With respect to the output $y_2(t)$, the system is observable in all subspaces of the state space .

$\overline{OB}$(II, 5555) - the system is a proper observable system in all subspaces of dimension two of the state space .

$\overline{OB}$(III, 5555) - the system is not a normal observable system.

$\overline{OB}$(I, 6666) - the system is not selectively uniformly observable with respect to the $y_1(t)$ output component, but is selectively uniformly observable with respect to the $y_2(t)$ output component.

$\overline{OB}$(II, 6666) - it is a proper observable system.

$\overline{OB}$(III, 6666) - it is not a normal observable system.

6.4. Necessary and Sufficient Conditions for State Invariance. For certain types of selective state invariance, the necessary and sufficient conditions are known [f.c. 145]. However, since in the existing literature on invariance reference is usually made to a somewhat different representation of dynamical systems, the essential proofs of the theorems will be given below.

Theorem $\overline{SI}(I, 4444)$. The system (5.2) is selectively uniformly i-th state invariant with respect to the j-th component of the input (disturbance) $\underline{u}(t)$, $j = 1, 2, \ldots, r$, if and only if

$$\det {}^{i}\Gamma^{j}(s) = 0$$

where the matrix ${}^{i}\Gamma^{j}(s)$ is obtained from the matrix $\Gamma(s) = (Is-A)$ by replacing its i-th column with the j-th column of the matrix B.

Proof. Let $\underline{u}_j^*(t) = [0, \ldots, 0, \underline{u}_j(t), 0 \ldots 0]$ be an r-component input vector with the j-th element being equal to the j-th element of the vector $\underline{u}(t)$ and zeroes elsewhere (here $u_j(t)$ is usually some disturbance input; see also p. 68). Substituting $\underline{u}_j^*(t)$ into the equation (5.2), we get

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}_j^*(t) = A\,\underline{x}(t) + \underline{b}_{\cdot j}\,u_j(t)$$

Taking the Laplace transform of the above expression, we can write

$$(sI - A)\,\underline{x}(s) = \underline{b}_{\cdot j}\,u_j(s) + \underline{r}(s)$$

which is a shorthand notation for the set of equations

$$(s - a_{11})\,x_1 - a_{12}\,x_2 - \ldots. - a_{1n}\,x_n = b_{1j}\,u_j(s) + x_1(0)$$

$$- a_{21}\,x_1 + (s - a_{22})\,x_2 - \ldots - a_{2n}\,x_n = b_{2j}\,u_j(s) + x_2(0)$$

$$- a_{n1} x_1 - a_{n2} x_2 - \dots + (s - a_{nn}) x_n = b_{nj} u_j(s) + x_n(0)$$

where $x_1(0), x_2(0), \dots, x_n(0)$ represent the initial conditions.

This is a system of algebraic equations which we can solve for any of the x's, i.e.

$$x_i(s) = \frac{\det \begin{bmatrix} (s - a_{11}) & -a_{12} & \dots & b_{1j}u_j(s) + x_1(o) & -- & - a_{1n} \\ -a_{21} & (s - a_{22}) & \dots & b_{2j}u_j(s) + x_2 & -- & - a_{2n} \\ ------ & ------ & ------ & ------ & ------ & ------ \\ ------ & ------ & ------ & ------ & ------ & ------ \\ -a_{n1} & -a_{n2} & \dots & b_{nj}u_j(s) + x_n(o) & \dots & (s - a_{nn}) \end{bmatrix}}{\det \begin{bmatrix} (s - a_{11}) & -a_{12} & -a_{13} & ------ & -a_{1n} \\ -a_{21} & (s-a_{22}) & -a_{23} & ------ & -a_{2n} \\ -a_{31} & -a_{32} & (s-a_{33}) & ------ & -a_{3n} \\ ------ & ------ & ------ & ------ & ------ \\ -a_{n1} & -a_{n2} & -a_{n3} & ------ & (s-a_{nn}) \end{bmatrix}}$$

Using the notation which has been introduced earlier, we can write

$$x_i(s) = \frac{\det {}^i\Gamma^j(s)}{\det \Gamma(s)} u_j(s) + \frac{\det {}^i\Gamma^{\underline{x}(o)}(s)}{\det \Gamma(s)} \tag{6.10}$$

where ${}^i\Gamma^j(s)$ stands for the matrix $\Gamma(s) = [sI - A]$ with its i-th row replaced by the j-th column of the matrix B, and ${}^i\Gamma^{\underline{x}(o)}(s)$ stands for the matrix $\Gamma(s)$ with its i-th row replaced by the vector $\underline{x}(o)$ representing the initial conditions.

Since, in general, $\det \Gamma(s) \neq 0$, it is clear that the necessary and sufficient condition that the component $x_i(t)$ of the state vector $\underline{x}(t)$ would not be affected by the (disturbance) input $u_j(t)$ is that $\det {}^i\Gamma^j(s) = 0$ in the Equation (6.10).

It is important to note that the statement "$x_i(t)$ is invariant with respect to the input $u_j(t)$" refers to the invariance of the steady state portion of the solution. The transient response depends, in general, on the initial conditions, and thus may depend indirectly also on the input $u_j(t)_{[t,\ t_o]}$ for $t < t_o$. This is clearly reflected in the Equation (6.10).

The necessary and sufficient conditions for selective invariance can also be expressed in terms of the controllability matrix as follows:

Corollary $\overline{\text{SI}}$(I, 4444). The linear time-invariant system (5.2) is selectively uniformly i-th state component invariant with respect to the j-th input component (disturbance) $u_j(t)$ if and only if all elements of the i-th row of the matrix

$$Q_j = [\underline{b}_{\cdot j} \vdots A\,\underline{b}_{\cdot j} \vdots \ldots \vdots A^{n-1}\,\underline{b}_{\cdot j}]$$

where $\underline{b}_{\cdot j}$ is the j-th column vector of the matrix B, are equal to zero.

Proof. With $\underline{u}_j(t) = [0, \ldots, 0, u_j(t), 0 \ldots, 0]^T$ we had [see proof of Theorem $\overline{SC}$(I, 4444)]

$$x_i(t) = [e^{At}]_{i\cdot}\, \underline{x}_o \tau \int_o^t [e^{A(t-\tau)}]_{i\cdot}\, \underline{b}_{\cdot j} u_j(\tau)\, d\tau$$

$$= [e^{At}]_{i\cdot}\, \underline{x}_o + \sum_{a=o}^{n=1} [A^{\alpha}]_{i\cdot}\, \underline{b}_{\cdot j}\, g_{\alpha}(t)$$

where

$$g_{\alpha}(t) = \int_o^t f_{\alpha}(\tau)\, u_j(\tau)\, d\tau$$

In order that the steady state portion of the response $x_i(t)$ would not depend on the input $u_j(t)$, it is necessary (and also sufficient) that

$$\sum_{\alpha=o}^{n-1} [A^{\alpha}]_{i\cdot}\, \underline{b}_{\cdot j}\, g_{\alpha}(t) = 0$$

for arbitrary values of $g_{\alpha}(t)$. For this to be the case it is necessary that the coefficients of $g_{\alpha}(t)$ be identically zero, i.e.

$$[A^{\alpha}]_{i\cdot}\, \underline{b}_{\cdot j} = 0 \qquad \text{for all } \alpha = 1, 2, \ldots, n-1$$

The latter requirement is identical with the requirement that the i-th row of the matrix

$$Q_j = [\underline{b}_{\cdot j} \vdots A\, \underline{b}_{\cdot j} \vdots \ldots. \vdots A^{n-1}\, \underline{b}_{\cdot j}]$$

contains elements which are all equal to zero. Q.E.D.

Example 17. Consider the system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \underline{u}(t) .$$

This system is $x_1(t)$ invariant with respect to $u_1(t)$ since

$$\det {}^1\Gamma^1(s) = \det \begin{bmatrix} 0 & 0 & 0 \\ 1 & s-1 & -1 \\ 1 & 0 & s-1 \end{bmatrix} = 0$$

It is also $x_3(t)$ - invariant with respect to the input $u_2(t)$ since

$$\det {}^3\Gamma^2(s) = \det \begin{bmatrix} s-1 & 0 & 1 \\ 0 & s-1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Using the criteria of the corollary $\overline{\text{SI}}$(I, 4444), we see that the first row of the matrix

$$Q_1 = [\underline{b}_{\cdot 1} \vdots A\,\underline{b}_{\cdot 1} \vdots A^2\,\underline{b}_{\cdot 1}] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

has all zeroes as well as the third row of the matrix

$$Q_2 = [\underline{b}_{\cdot 2} \vdots A\,\underline{b}_{\cdot 2} \vdots A^2\,\underline{b}_{\cdot 2}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so that we have the same result applying either criterion, as was to be expected.

Now, it is clear that a component $x_i(t)$ of the state vector $\underline{x}(t)$ is invariant with respect to some input $\underline{u}(t)$ if and only if it is invariant with respect to each input component individually. Hence there immediately follows

Theorem $\overline{\text{SI}}$(II, 4444). The system (5.2) is proper i-th state invariant if and only if $\det {}^{i}\Gamma^{j}(s) = 0$ for all $j = 1, 2, \ldots, r$, or alternatively, if and only if all the elements of the i-th rows of the matrix

$$Q_j = [\underline{b}_{\cdot j} \vdots A\,\underline{b}_{\cdot j} \vdots \ldots \vdots A^{n-1}\,\underline{b}_{\cdot j}]$$

are zeroes for all $j = 1, 2, \ldots, r$.

Theorem $\overline{\text{SI}}$(III, 4444). The system (5.2) is normal i-th state invariant if and only if it is proper i-th state invariant.

Furthermore, the extension of the criteria of invariance to more than one component of the state vector is also simpler than in the case of controllability or observability. It has been shown in the preceding sections that controllability or observability of some individual components of the state or output vector does not imply that they are controllable (observable) simultaneously in some subspace $\tilde{\Sigma}(R^k)$. But if a system (5.2) is selectively i-th state invariant and selectively j-th state invariant with regard to some input $u_k(t)$, then it is also true that the system is invariant in the subspace of Σ generated by the vectors $[0, \ldots, 0, x_i(t), 0, \ldots, 0]^T$ and $[0, \ldots, 0, x_j(t), 0, \ldots, 0]^T$. This leads us to the

Theorem $\overline{\text{SI}}$(I, 5555). The system is selectively uniformly state invariant in the subspace $\tilde{\Sigma}$ of the state space Σ, if and only if it is selectively uniformly state invariant in all the components of the state vector $\underline{x}(t)$ which have non-zero projections in the subspace $\tilde{\Sigma}$, or alternatively, if and only if the n x n matrix

$$E_k \, Q_j = [E_k \, \underline{b}_{\cdot j} \vdots E_k \, A \, \underline{b}_{\cdot j} \vdots \ldots \vdots E_k \, A^{n-1} \, \underline{b}_{\cdot j}]$$

where E_k is the projector of Σ on $\tilde{\Sigma}$, is a zero matrix.

Theorem $\overline{\text{SI}}$(II, 5555). The system (5.2) is a proper state-invariant system in the subspace $\tilde{\Sigma}$ of the state space Σ if and only if the n x n matrix

$$E_k \, Q = [E_k \, B \vdots E_k \, A \, B \vdots \ldots \vdots E_k \, A^{n-1} \, B]$$

with E_k defined as above, is a zero matrix.

<u>Theorem $\overline{\mathrm{SI}}$(III, 5555)</u>. The system (5.2) is a normal state-invariant system in the subspace $\tilde{\Sigma}$ of the state space Σ if and only if it is proper state-invariant in that subspace.

A natural consequence of what was stated in the preceding portion of this section is also the following

<u>Theorem SI(I, 666)</u>. The system (5.2) is selectively uniformly state-invariant with respect to the j-th input (disturbance), $j = 1, 2, \ldots, r$, if and only if the $n \times n$ matrix

$$Q_j = [\underline{b}_{\cdot j} \vdots A\, \underline{b}_{\cdot j} \vdots \ldots \vdots A^{n-1}\, \underline{b}_{\cdot j}]$$

is a zero matrix, i.e., if $\underline{b}_{\cdot j}$ is a zero vector.

<u>Theorem SI(II, 6666)</u>. The system (5.2) is a proper state-invariant system if and only if the composite $n \times nk$ matrix

$$Q = [B \vdots A\, B \vdots \ldots. \vdots A^{n-1}\, B]$$

is a zero matrix, i.e., if B is a zero matrix.

<u>Theorem $\overline{\mathrm{SI}}$(III, 6666)</u>. The system (5.2) is a normal state-invariant if and only if it is proper state invariant.

<u>Example 18</u>. Consider the system given by the equations

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \underline{u}(t)$$

We find

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} ; \quad Q_2 = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 3 \\ 1 & 1 & 7 \end{bmatrix} ; \quad Q_3 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 6 & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

By Theorem $\overline{SI}$(II, 4444), the system is proper $x_1(t)$ and $x_3(t)$ invariant with respect to the input component $u_1(t)$ as well as input component $u_3(t)$. (1-st and 3-rd rows of Q_1 and Q_2 contain only zero elements).

By Theorem $\overline{SI}$(I, 5555), the system is selectively uniformly invariant in the subspace represented by the (x_1, x_3) - plane both with respect to the input $u_1(t)$ and $u_3(t)$ since

$$E_2\, Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$E_2\, Q_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 \\ 3 & 6 & 12 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By Theorem $\overline{SI}$(II, 5555), there exists no subspace of the state space Σ in which the system is proper state-invariant.

By Theorem $\overline{SI}$(I, 6666), the system is not selectively uniformly invariant in either $x_1(t)$, $x_2(t)$, or $x_3(t)$ components with respect to any of the input components.

By Theorem $\overline{SI}(II, 6666)$, the system is not a proper state-invariant system, since

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 4 & 0 \\ 1 & 0 & 3 & 2 & 1 & 6 & 4 & 3 & 12 \\ 6 & 1 & 0 & 0 & 4 & 0 & 0 & 7 & 0 \end{bmatrix} \neq [0]$$

6.5. <u>Necessary and Sufficient Conditions for Output Invariance</u>.

Criteria for output invariance are derivable either directly from those of state invariance or by the methods paralleling the derivation of the latter. Therefore the proofs will be omitted in this section completely.

<u>Theorem $\overline{OI}(I, 4444)$</u>. The system (5.2) is selectively uniformly i-th output invariant with respect to the j-th component of the input (disturbance) $\underline{u}(t)$, $j = 1, 2, \ldots, r$, if and only if

$$f_{ik} \det {}^{k}\Gamma^{j}(s) = 0 \qquad \text{for all } k = 1, 2, \ldots, n$$

and

$$g_{ij} = 0$$

where ${}^{k}\Gamma^{j}(s)$ is a matrix as defined on page 78, g_{ij} is the (i,j)-th element of the matrix G, and f_{ik} is the (i,k)-th element of the matrix F.

<u>Corollary $\overline{OI}(I, 4444)$</u>. The system (5.2) is selectively uniformly i-th output invariant with respect to the j-th component of the input

(disturbance) $\underline{u}(t)$, $j = 1, 2, \ldots, r$, if and only if all elements of the i-th row of the $p\times(n+1)$ matrix

$$W_j = [F\,\underline{b}_{.j} \vdots F\,A\,\underline{b}_{.j} \vdots \ldots \vdots F\,A^{n-1}\,\underline{b}_{.j} \vdots \underline{g}_{.j}]$$

are equal to zero. Here $\underline{b}_{.j}$ is the j-th column vector of the matrix B and $\underline{g}_{.j}$ is that of the matrix G.

Theorem $\overline{OI}$(II, 4444). The system (5.2) is proper i-th output invariant if and only if

$$f_{ik} \det {}^{k}\Gamma^{j}(s) = 0 \quad \text{for all } k = 1, 2, \ldots, n \text{ and all } j = 1, 2, \ldots, r$$

and

$$g_{ij} = 0 \quad \text{for all } j = 1, 2, \ldots, r$$

with f_{ik}, g_{ij}, and ${}^{k}\Gamma^{j}(s)$ defined as above, or alternatively, if and only if all the elements of the i-th rows of the matrices

$$W_j = [F\,\underline{b}_{.j} \vdots F\,A\,\underline{b}_{.j} \vdots \ldots\ldots \vdots F\,A^{n-1}\,\underline{b}_{.j} \vdots \underline{g}_{.j}]$$

are equal to zeroes for all $j = 1, 2, \ldots, r$.

Theorem $\overline{OI}$(III, 4444). The system (5.2) is normal i-th output invariant if and only if it is proper i-th output invariant.

Theorem $\overline{OI}$(I, 5555). The system is selectively uniformly output-invariant in the subspace R^q of the output space R^p, $q < p$, if and only if it is selectively uniformly output-invariant in all the components of the output vector $\underline{y}(t)$ which have non-zero projections in the subspace R^q, i.e., if and only if the p x(n + 1) matrix

$$E_k W_j = [E_k F \underline{b}_{\cdot j} \vdots E_k F A \underline{b}_{\cdot j} \vdots \dots \vdots E_k F A^{n-1} \underline{b}_{\cdot j} \vdots E_k \underline{g}_{\cdot j}]$$

where E_k is the projector of R^p on R^q, is a zero matrix.

<u>Theorem $\overline{\text{OI}}$(II, 5555)</u>. The system (5.2) is a proper output-invariant system in the subspace R^q of the output space R^p if and only if the px(n + 1) matrices

$$E_k W_j = [E_k F \underline{b}_{\cdot j} \vdots E_k F A \underline{b}_{\cdot j} \vdots \dots \vdots E_k F A^{n-1} \underline{b}_{\cdot j} \vdots E_k \underline{g}_{\cdot j}]$$

with E_k defined as above, are zero matrices for all $j = 1, 2, \dots, r$.

<u>Theorem $\overline{\text{OI}}$(III, 5555)</u>. The system (5.2) is a normal output-invariant system in the subspace R^q of the output space R^p if and only if it is proper output-invariant in that subspace.

<u>Theorem $\overline{\text{OI}}$(I, 6666)</u>. The system (5.2) is selectively uniformly output-invariant with respect to the j-th input (disturbance), $j = 1, 2, \dots, r$, if and only if the px(n + 1) matrix

$$W_j = [F \underline{b}^j \vdots F A \underline{b}_{\cdot j} \vdots F A^2 \underline{b}_{\cdot j} \vdots \dots \vdots F A^{n-1} \underline{b}_{\cdot j} \vdots \underline{g}_{\cdot j}]$$

is a zero matrix.

<u>Theorem $\overline{\text{OI}}$(II, 6666)</u>. The system (5.2) is a proper output-invariant system if and only if the composite p x r(n + 1) matrix

$$W = [F\,B \vdots F\,A\,B \vdots F\,A^2\,B \vdots \ldots . \vdots F\,A^{n-1}\,B \vdots D]$$

is a zero matrix, i.e., if B and G are zero matrices.

Theorem $\overline{\mathrm{OI}}$(III, 6666). The system (5.2) is normal output-invariant if and only if it is proper output-invariant.

Example 19. Consider the system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}(t)$$

Here we have

$$\begin{aligned} W_1 &= [F\,\underline{b}_{\cdot 1} \vdots F\,A\,\underline{b}_{\cdot 1} \vdots F\,A^2\,\underline{b}_{\cdot 1} \vdots \underline{g}_{\cdot 1}] \\ &= \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \end{aligned}$$

$$\begin{aligned} W_2 &= [F\,\underline{b}_{\cdot 2} \vdots F\,A\,\underline{b}_{\cdot 2} \vdots F\,A^2\,\underline{b}_{\cdot 2} \vdots \underline{g}_{\cdot 2}] \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 4 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} W &= [F\,B \vdots F\,A\,B \vdots F\,A^2\,B \vdots G] \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 & 0 & 1 \end{bmatrix} \end{aligned}$$

By Theorem $\overline{OI}$(I, 4444), the above system is selectively uniformly $y_2(t)$ output invariant with respect to the second component of the input vector, $u_2(t)$. This is the only selective output invariance for this system.

By Theorem $\overline{OI}$(II, 4444), it is neither $y_1(t)$ nor $y_2(t)$ output invariant proper.

By Theorem $\overline{OI}$(III, 4444), it is neither $y_1(t)$ nor $y_2(t)$ output invariant normal.

By Theorem $\overline{OI}$(I, 6666), the system is neither selectively uniformly output invariant with respect to the input $u_1(t)$ nor with respect to the input $u_2(t)$.

By Theorem $\overline{OI}$(II, 6666), it is not a proper output-invariant system and

By Theorem $\overline{OI}$(III, 6666), it is neither a normal output invariant system.

CHAPTER VII

RELATION BETWEEN CONTROLLABILITY AND INVARIANCE

We have already seen that the set of all controllable states of the system (5.2) is a linear vector subspace of the state space Σ . Denote this set of all controllable states by $R(Q)$ and the orthogonal complement of this set by $R(Q)^{\perp}$. Thus we know that the direct sum of these is

$$\Sigma = R(Q) \oplus R(Q)^{\perp}$$

We also know that the set of controllable states, $R(Q)$, is invariant under transformation A .

We shall now state and prove the following.

<u>Theorem (7.1)</u>. Consider the system (5.2) with $R(Q)$ the set of controllable states. Then the system is state-invariant [i.e., $\overline{SI}$(II, 5555)] in the subspace $R(Q)^{\perp}$ for all $\underline{x} \in R(Q)^{\perp}$.

<u>Proof</u>. Let $\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k\}$ be the basis of the linear vector space $R(Q)$ and $\{\underline{x}_{kt1}, \ldots, \underline{x}_n\}$ the basis of the orthogonal complement $R(Q)^{\perp}$. It has been shown that the column vectors of the matrix

$$Q = [B \vdots A B \vdots A^2 B \vdots \ldots\ldots \vdots A^{n-1} B]$$

span the space of controllable states $R(Q)$. Hence, any column vector of the matrix Q can be expressed as a linear combination of the basis vectors $\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k\}$. Let E_k be the orthogonal projector on the space $R(Q)^{\perp}$, then

$$E_k \, \underline{x} = 0 \quad \text{for all} \quad \underline{x} \in R(Q)$$

Since every column vector of the matrix Q is an element of the vector space R, it follows that

$$E_k \, Q = [0]$$

Hence, by Theorem $\overline{\text{SI}}$(II, 5555), the system is invariant in the subspace $R(Q)^{\perp}$.

Example 20. Consider the system, the state equations of which are

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

For this system, the matrix

$$Q = [B \,\vdots\, A\,B \,\vdots\, A^2\,B] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and hence the set of controllable states is the one dimensional linear vector space generated by the vector $[1 \;\; 1 \;\; 1]^T$. Let it be the basis. The orthogonal complement $R(Q)^{\perp}$ to this set of controllable states

$R(Q)$ is the set of vectors in the two dimensional vector space with the basis (which, of course, is not unique)

$$\{\underline{x}_2,\ \underline{x}_3\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

The projector of the state space Σ on the space $R(Q)^{\perp}$ is the operator

$$E = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

and we can easily verify that the product

$$E\,Q = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is a zero matrix. Hence, the system is invariant in the two dimensional vector space spanned by the vectors $\underline{x}_2 = [-2 \;\; 1 \;\; 1]^T$ and $\underline{x}_3 = [0 \;\; 1 \;\; -1]^T$, i.e., in the plane which goes through the origin and is perpendicular to the vector $\underline{x}_1 = [1 \;\; 1 \;\; 1]^T$.

Two dynamical systems S and S^* are said to be <u>equivalent</u> if there is a nonsingular $n \times n$ matrix P with constant elements such that

$$P^{-1}\,\underline{x}(t) = \underline{z}(t)$$

where $\underline{x}(t)$ is the state variable of S and $\underline{z}(t)$ is the state variable of S^*. It has been shown [15] that the concepts of controllability and observability are intrinsic, in the sense that they are preserved under equivalence. More specifically, if P is a nonsingular $n \times n$ matrix and $\underline{x}(t) = P^{-1}\,\underline{x}(t)$, then the system

$$\dot{\underline{x}}(t) = \underline{f}\,[\underline{x}(t), \underline{u}(t), t]$$

$$\underline{y}(t) = \underline{g}\,[\underline{x}(t), \underline{u}(t), t]$$

is controllable (observable) if and only if the equivalent system

$$\dot{\underline{z}}(t) = P^{-1}\,\underline{f}\,[P\underline{z}(t)\,, \underline{u}(t), t]$$

$$\underline{y}(t) = g\,[P\underline{z}(t)\,, \underline{u}(t), t]$$

is controllable (observable). However, it is important to note that equivalence reflects only on the dimensionality of the controllable and invariant state spaces: if two systems S and S^* are equivalent, then the dimension of the space of controllable (observable) states of the system S is equal to the dimension of the space of controllable (observable) states of the system S^*. By the same token, the direct complements of the space of controllable states of these two systems are also of the same dimensionality. On the other hand, similarity transformation does affect selective controllability and invariance in the sense that the components of the state vector which are controllable or invariant in the system S might not be those which are controllable or invariant in system S^*.

Among the systems S^* which are equivalent to the given system S, of particular interest is the system in which the basis of the sub-space of the controllable states is chosen from among vectors co-linear with the coordinate axes, i.e., a subset of the natural basis. It is clear that in such a case the basis of the orthogonal complement of the set of controllable states, i.e., the basis of the set of invariant states, consists also of vectors co-linear with coordinate axes. We can obtain such a system as follows.

Let $\Sigma = R(Q) + R(Q)^{\perp}$ and let $\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k\}$ be the basis for $R(Q)$. Let $\{\underline{x}_{k+1}, \ldots, \underline{x}_n\}$ be the basis of $R(Q)^{\perp}$. Then $\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\}$ is the basis of Σ. Let $P : \Sigma \to \Sigma$ be a linear transformation of rank n corresponding to the above basis. If the system S is given by the equations

$$\begin{aligned} \dot{\underline{x}}(t) &= A\,\underline{x}(t) + B\,\underline{u}(t) \\ \underline{y}(t) &= F\,\underline{x}(t) + G\,\underline{u}(t) \end{aligned} \tag{5.2}$$

then the equivalent system S^{**}

$$\begin{aligned} \dot{\underline{z}}(t) &= A^*\,\underline{z}(t) + B^*\,\underline{u}(t) \\ \underline{y}(t) &= F^*\,\underline{z}(t) + G\,\underline{u}(t)\ , \end{aligned} \tag{7.11}$$

obtained from S by substituting

$$\underline{x}(t) = P\, \underline{z}(t) ,$$

has matrix representations of the linear transformations A^* and B^* in the form

$$A^* = \begin{bmatrix} A_{11}^* & A_{12}^* \\ 0 & A_{22}^* \end{bmatrix} ; \quad B^* = \begin{bmatrix} B_1^* \\ 0 \end{bmatrix}$$

where A_{11}^* is a k x k matrix, A_{12}^* is k x (n-k), A_{22}^* is (n-k)x(n-k), and B_1^* is k x k. In other words, equation (7.11) can be partitioned into two vector equations

$$\dot{\underline{z}}_1(t) = A_{11}^*\, \underline{z}_1(t) + A_{12}^*\, \underline{z}_2(t) + B_1^*\, \underline{u}(t)$$

$$\dot{\underline{z}}_2(t) = A_{22}^*\, \underline{z}_2(t)$$

where $\underline{z}(t) = [\underline{z}_1(t), \underline{z}_2(t)]^T$, and $[\underline{z}_1(t), \underline{0}]^T \in R(Q)$, $[\underline{0}, \underline{z}_2(t)]^T \in R(Q)^{\perp}$. It is clear that for this system the components $z_{k+1}(t), \ldots, z_n(t)$ of the state vector $\underline{z}(t)$ are invariant, i.e., are not affected by the input $\underline{u}(t)$.

The proof that the transformation $\underline{x}(t) = P\, \underline{z}(t)$ results in the above representation of the system S^* equivalent to the system S is given by Zadeh [291], and will not be repeated here. Instead, it will be illustrated by an example.

Example 21. Consider the system of example 20, with $\{[1\ \ 1\ \ 1]^T\}$ the basis of $R(Q)$ and $\{[-2\ \ 1\ \ 1]^T, [0\ \ 1\ \ -1]^T\}$ the basis of $R(Q)^\perp$. Hence

$$P = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

and

$$P^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Taking

$$\underline{x}(t) = P\,\underline{z}(t)$$

we obtain

$$\dot{\underline{z}}(t) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \underline{z}(t)$$

(equation concluded on following page.)

$$+ \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) = \begin{bmatrix} 1 & 1 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \underline{z}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

which can be partitioned into two vector equations

$$\dot{\underline{z}}_1(t) = [1]\ \underline{z}_1(t) + [1 \quad \tfrac{1}{3}]\ \underline{z}_2(t) + [1]\ u(t)$$

$$\dot{\underline{z}}_2(t) = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & 1 \end{bmatrix} \underline{z}_2(t)$$

where

$$\underline{z}_1(t) = [z_1(t)]$$

$$\underline{z}_2(t) = \begin{bmatrix} z_2(t) \\ z_3(t) \end{bmatrix}$$

and

$$\underline{z}(t) = \begin{bmatrix} \underline{z}_1(t) \\ \underline{z}_2(t) \end{bmatrix}$$

The formal similarities of criteria of controllability and invariance imply that there should be some kind of duality relation between controllable and invariant systems. And, indeed, there is such a relation, as it can be seen from the following.

<u>Theorem (7.2)</u>. Consider the linear time invariant system

$$\begin{aligned} \dot{\underline{x}}(t) &= A\, \underline{x}(t) + B\, \underline{u}(t) \\ \underline{y}(t) &= F\, \underline{x}(t) + G\, \underline{u}(t) \end{aligned} \tag{5.2}$$

which is controllable in the state space $R(Q)$ of dimension k. Then the dual system

$$\begin{aligned} \hat{\underline{x}}(t) &= \tilde{A}\, \hat{\underline{x}}(t) + \tilde{B}\, \hat{\underline{u}}(t) \\ \hat{\underline{y}}(t) &= F\, \hat{x}(t) + G\, \hat{\underline{u}}(t) \end{aligned} \tag{7.12}$$

in which $\tilde{A} = (I-E_k)\, A$, E_k is the orthogonal projector on $R(Q)$ along $R(Q)^{\perp}$, and $\tilde{B} = \{\underline{x}_{k+1}, \ldots, \underline{x}_n\}$ is a matrix corresponding to some basis in $R(Q)^{\perp}$, is invariant in the subspace $R(Q)$ of the state space Σ of dimension k.

<u>Proof</u>. The system S is controllable in the subspace $R(Q)$ of the state space Σ by assumption. Then, by Theorem (7.1), it is invariant in the subspace $R(Q)^{\perp}$, i.e., in the orthogonal complement of the subspace $R(Q)$. If E_k is the projector on the subspace $R(Q)$, then $I-E_k$, where I is the identity matrix, is the projector on the subspace $R(Q)^{\perp}$ [251].

Assume now that the system $\hat{S}$ is not invariant in the subspace

R(Q). Then by Theorem SI(II, 5555), the matrix

$$E_k \tilde{Q} = E_k [\tilde{B} \vdots (I-E_k) A \tilde{B} \vdots ((I-E_k)A)^2 \tilde{B} \vdots .. \vdots ((I-E_k)A)^{n-1} \tilde{B}]$$

is not a zero matrix. Multiplying through the right hand side of the expression, we get

$$E_k \tilde{Q} = [E_k \tilde{B} \vdots E_k(I-P_k) A \tilde{B} \vdots ..... \vdots E_k((I-E_k)A)^{n-1} \tilde{B}]$$

Now, $E_k \tilde{B}$ is a zero matrix, because the column vectors $\underline{\tilde{b}}_{\cdot j}$ of the matrix $\tilde{B}$ are elements of $R(Q)^{\perp}$ by assumption, and so $E_k \underline{\tilde{b}}_{\cdot j} = \underline{0}$ for all $j = k + 1, ..., n$. The matrices $E_k(1-E_k) A \tilde{B}, ..., E_k((I-E_k)A)^{n-1} \tilde{B}$ are also zero matrices because

$$E_k (I-E_k) = E_k - E_k^2 = E_k - E_k = 0$$

Hence the composite matrix $E_k\tilde{Q}$ is a zero matrix, which is a contradiction, and therefore the system $\hat{S}$ is invariant in the subspace R(Q) of the state space Σ .

We shall illustrate this important relationship by the following

Example 22. Consider again the system given by the state equations

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

We have seen in Example 21, that the system is controllable in the subspace $R(Q)$ generated by the vector $\underline{x} = [1 \;\; 1 \;\; 1]^T$

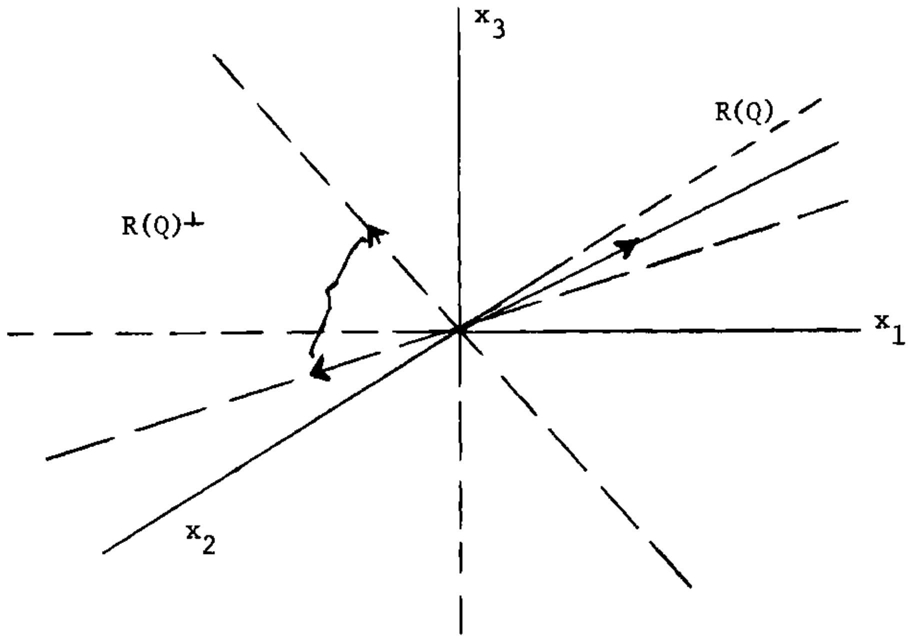

Figure 1.

One choice of a basis in $R(Q)^{\perp}$. is $\{[-2 \;\; 1 \;\; 1]^T \,, \;\; [0 \;\; 1 \; -1]^T\}$. The projector on $R(Q)$ along $R(Q)^{\perp}$ is

$$E_1 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Hence, by the preceding theorem, the system

$$\dot{\hat{\underline{x}}}(t) = (I-E_k)\, A\, \hat{\underline{x}}(t) + \tilde{B}\, \underline{u}(t) = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{\underline{x}}(t) + \begin{bmatrix} -2 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \underline{u}(t)$$

$$= \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \hat{\underline{x}}(t) + \begin{bmatrix} -2 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \underline{u}(t)$$

is invariant in the subspace $R(Q)$ with respect to any input $\underline{u}(t)$. The spaces $R(Q)$ and $R(Q)^{\perp}$ are shown on the diagram Figure 1.

The Theorem 7.2 can be formulated in a somewhat different fashion which can serve as a handy criteria for controllability and invariance. Namely:

Corollary (7.2). The linear constant parameter dynamical system

$$\begin{aligned} \dot{\underline{x}}(t) &= A\, \underline{x}(t) + B\, \underline{u}(t) \\ \underline{y}(t) &= F\, \underline{x}(t) + G\, \underline{u}(t) \end{aligned} \tag{5.2}$$

is controllable (invariant) in the subspace $R(Q)$ of the state space Σ if and only if the dual system

$$\dot{\hat{\underline{x}}}(t) = \tilde{A}\,\hat{\underline{x}}(t) + \tilde{B}\,\hat{\underline{u}}(t)$$
$$\hat{\underline{y}}(t) = F\,\hat{\underline{x}}(t) + G\,\hat{\underline{u}}(t) \tag{7.12}$$

is invariant (controllable) in the same subspace $R(Q)$, where $\tilde{A}$ and $\tilde{B}$ are defined as in Theorem (7.2).

There is also an important relationship between state controllability and output invariance as the following theorem shows.

<u>Theorem (7.3)</u>. Consider the constant parameter dynamical system.

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t)$$
$$\underline{y}(t) = F\,\underline{x}(t) + G\,\underline{u}(t) \tag{5.2}$$

If the system is controllable in the subspace $R(Q)$ of the state space Σ, then the system

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t)$$
$$\underline{y}(t) = \tilde{B}^T\underline{x}(t) \tag{7.13}$$

is a proper output invariant system where $\tilde{B}$ is a matrix corresponding to some basis $(\underline{x}_{k+1}, \ldots, \underline{x}_n)$ of the subspace $R(Q)^{\perp}$.

Proof follows immediately from Theorem $\overline{\mathrm{OI}}$(II, 6666), since

$$H = [\tilde{B}^T B \;\vdots\; \tilde{B}^T AB \;\vdots\; \ldots \;\vdots\; \tilde{B}^T A^{n-1} B \;\vdots\; D]$$

is a zero matrix because $\tilde{B}^T\tilde{B}$ is a zero matrix due to the orthogonality of column vectors of $\tilde{B}$ with respect to B, R(Q) is invariant subspace under transformation A, and D is a zero matrix by assumption.

Example 23. For the state equations of example 20, the system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$\underline{y}(t) = \begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \underline{x}(t)$$

is a proper output invariant system.

A direct consequence of the preceding discussions is also the following theorem, which, regardless of its simplicity, has important practical applications.

Theorem (7.4). Consider the system (5.2). If the system is not selectively uniformly i-th state (or i-th output) controllable (invariant) with respect to some input $u_j(t)$, $j = 1, 2, \ldots, r$, then it is selectively uniformly i-th state (or i-th output) invariant (controllable) with respect to the same input $u_j(t)$.

Proof follows immediately from Corollary $\overline{SC}$(I, 4444) and $\overline{SI}$(I, 4444) for state controllability and invariance, and from Corollaries $\overline{OC}$(I, 4444) and $\overline{OI}$(I, 4444) for output controllability and invariance, respectively.

Note that this theorem cannot be generalized to non-selective controllability or invariance, for instance, to controllability of some set of state (or output) components with respect to single or several components of the input vector.

CHAPTER VIII

RELATION BETWEEN OBSERVABILITY AND INVARIANCE

Many relations between observability follow directly from the duality theorem (6.1) linking observability and controllability and the theorem (7.2) linking controllability to invariance.

We have already noted (see p. 84) that the state space Σ can be decomposed into a direct sum of the subspace of observable states $R(H)$ and the subspace of unobservable states $R(H)^{\perp}$, both of which are linear subspaces of Σ, so that

$$\Sigma = R(H) \oplus R(H)^{\perp}$$

Furthermore, the subspace of unobservable states $R(H)^{\perp}$ is invariant under the linear transformation A.

Now, we have first of all the

Theorem (8.1). Consider the system Eq. (5.2), i.e.,

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t)$$

$$\underline{y}(t) = F\,\underline{x}(t) + G\,\underline{u}(t)$$

which is observable in some subspace $R(H)$ of the state space Σ. Then the system

$$\dot{\hat{\underline{x}}}(t) = -A^T \hat{\underline{x}}(t) + F^T \hat{\underline{u}}(t)$$

$$\hat{\underline{y}}(t) = B^T \hat{\underline{x}}(t) + G \hat{\underline{u}}(t)$$

is state-invariant in the subspace $R(H)^{\perp}$ of the state space Σ .

Proof follows immediately from the Theorem (6.1) and Theorem (7.2).

If the system (5.2) is unobservable in some subspace $R(H)^{\perp}$ of the state space Σ , then there exists an equivalent system for which the basis vectors of the space $R(H)^{\perp}$ coincide with some of the coordinate axes. It is again the question of selecting an appropriate basis and the procedure is analogous to that in the previous section.

Thus, let $\Sigma = R(H) + R(H)^{\perp}$ and let $\{\underline{x}_1, x_2, \ldots, \underline{x}_j\}$ be the basis for $R(H)$. We can extend this basis to form the basis $\{\underline{x}_1, \ldots, \underline{x}_n\}$ of the state space Σ with $\{\underline{x}_{j+1}, \ldots, \underline{x}_n\}$ being the basis of $R(H)^{\perp}$. If the system S is given by the equations

$$\dot{\underline{x}}(t) = A\, \underline{x}(t) + B\, \underline{u}(t)$$

$$\underline{y}(t) = F\, \underline{x}(t) + G\, \underline{u}(t)$$

and P is a linear transformation corresponding to the basis $\{\underline{x}_1, \ldots, \underline{x}_j, \underline{x}_{j+1}, \ldots, \underline{x}_n\}$, then by substituting

$$\underline{x}(t) = P\, \underline{z}(t)$$

we obtain an equivalent system S^*

$$\dot{\underline{z}}(t) = A^o \underline{z}(t) + B^o \underline{u}(t)$$

$$\underline{y}(t) = F^o \underline{z}(t) + G \underline{u}(t)$$

in which the matrices A^o and F^o have the form

$$A^o = \begin{bmatrix} A_{11}^o & 0 \\ A_{12}^o & A_{22}^o \end{bmatrix} ; F^o = [F_1^o \vdots 0],$$

where A_{11}^o is a j x j matrix, A_{12}^o is (n-j) x j, A_{22}^o is (n-j) x (n-j), and F^o is p x j matrix. Assuming $G = 0$, we can partition the above equation into two vector equations

$$\dot{\underline{z}}_1(t) = A_{11}^o \underline{z}_1(t) + B_1^o \underline{u}_1(t)$$

$$\dot{\underline{z}}_2(t) = A_{12}^o \underline{z}_1(t) + A_{22}^o \underline{z}_2(t) + B_2^o \underline{u}_2(t)$$

and

$$\underline{y}_1(t) = F_1^o \underline{z}_1(t)$$

$$\underline{y}_2(t) = \underline{0}$$

so that it is obvious that the system is output invariant with respect to the output components comprising the subvector $\underline{y}_2(t)$.

The proof of the above proposition is essentially the same as Zadeh's [29] proof for the decomposition of the state space into controllable subspace and its orthogonal complement, except that here it is the set of unobservable states $R(H)^{\perp}$ which is an invariant subspace under linear transformation A.

Finally, we have also the analogue of the duality theorem relating observability to invariance, i.e.,

Theorem (8.2). The linear time-invariant dynamical system Equation (5.2), i.e.,

$$\underline{\dot{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t)$$

$$\underline{y}(t) = F\,\underline{x}(t) + G\,\underline{u}(t)$$

is observable (invariant) in the subspace $R(H)$ of dimension j generated by the column vectors of the matrix $[F^T \vdots A^T F^T \vdots \ldots \vdots (A^T)^{n-1} F^T]$ if and only if the dual system

$$\dot{\underline{\tilde{x}}}(t) = \hat{A}\,\underline{\tilde{x}}(t) + \hat{B}\,\underline{\tilde{u}}(t)$$

$$\underline{\tilde{y}}(t) = \hat{F}\,\underline{\tilde{x}}(t) + G\,\underline{\tilde{u}}(t)$$

is invariant (observable) in the subspace $R(H)^{\perp}$. Here the matrix $\hat{A} = (E_k - I)\,A^T$, the matrix $\hat{B}$ is a n x (n-j) matrix corresponding to some basis $(\underline{x}_{j+1}, \ldots, \underline{x}_n)$ of the subspace $R(H)^{\perp}$.

Example 24. Consider the system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1 \quad 1] \; \underline{x}(t)$$

The observability matrix for this system is

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Hence, the set of observable states is the one dimensional space generated by the vector $[0 \quad 1 \quad 1]^T$. In order to transform the system into the canonical form, we choose as the basis of the orthogonal complement $R(H)^{\perp}$, i.e., of the space of unobservable states, the vectors $\{[1 \quad 0 \quad 0]^T , [0 \quad -1 \quad 1]^T\}$. Then the corresponding matrix is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$P^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

Substituting

$$\underline{x}(t) = P\,\underline{z}(t)$$

into our systems equations, we obtain

$$\dot{\underline{z}}(t) = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \times \begin{bmatrix} 0 & \underline{1} & \underline{1} \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \underline{z}(t)$$

$$+ \begin{bmatrix} 0 & \underline{1} & \underline{1} & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 1 \quad 1] \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \underline{z}(t)$$

or

$$\dot{\underline{z}}(t) = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 1 & 0 & -1 \\ -2 & 0 & -1 \end{array}\right] \underline{z}(t) \qquad \left[\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right] u(t)$$

$$y(t) = [2 \quad 0 \quad 0]\,\underline{z}(t)$$

This can be partitioned into vector equations

$$\underline{\dot{z}}_1(t) = [\,-1\,]\ \underline{z}_1(t) \qquad\qquad + [\,1\,]\ u(t)$$

$$\underline{\dot{z}}_2(t) = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \underline{z}_1(t) + \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \underline{z}_2(t)$$

$$y(t) = [\,2\,]\ \underline{z}_1(t) + \underline{0} \cdot \underline{z}_2(t)$$

where

$$\underline{z}_1(t) = [z_1(t)]\ ; \quad \underline{z}_2(t) = \begin{bmatrix} z_2(t) \\ z_3(t) \end{bmatrix}$$

CHAPTER IX

INTERACTION ANALYSIS

In formalizing the concept of interaction in multivariable or multidimensional systems, it is expedient to introduce the notion of "sets of inputs" $\{u\}$, "sets of states" $\{x\}$ and "sets of outputs" $\{y\}$ to indicate sets of components of the input, state and output vectors* . The purpose of this notation is to indicate that only certain components of the input, state or output vectors are being considered, without making any implications as to the behavior of or effect on other components of these vectors.

It is possible to classify interactions in multivariable or multidimensional systems into the following categories:

(1) Input-input interactions between some sets $\{u\}$.

(2) Input-state interactions between sets $\{u\}$ and $\{x\}$.

(3) Input-output interactions between $\{u\}$ and $\{y\}$.

(4) State-input interactions between $\{x\}$ and $\{u\}$.

(5) State-state interactions between some sets $\{x\}$.

(6) State-output interactions between $\{x\}$ and $\{y\}$.

(7) Output-input interactions between $\{y\}$ and $\{u\}$.

(8) Output-state interactions between $\{y\}$ and $\{S\}$.

(9) Output-output interactions between some sets $\{y\}$.

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* The main reason for this change of notation is that the terminology in terms of subspaces becomes very clumsy.

The question might be raised whether one should differentiate between, say, state-output and output-state interactions. It seems that such a differentiation is justifiable and useful because of the lack of commutativity property of interactions. As an illustration, consider systems shown in Figures 2 and 3. In the system shown in Figure 2 it is feasible to talk about output-state interaction between the sets x_1 and y_1, but there is no state-output interaction between those two sets. Similarly, in the system shown in Figure 3 one can talk about state-output interaction between the state set x_2 and output set y_1, but there is no output-state interaction. It is because of the inherent assumption of the unidirectional signal flow that the discrimination in the "direction" of interaction is required.

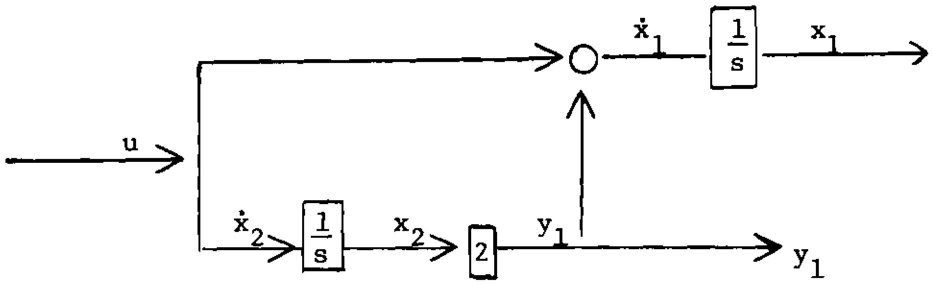

Figure 2.

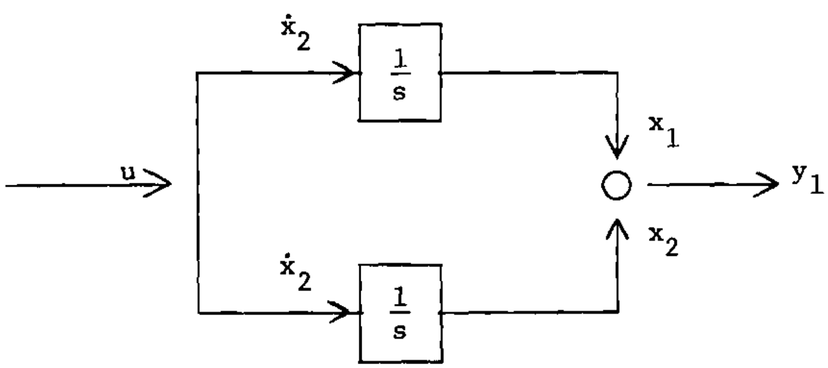

Figure 3.

9.1 Interaction Structures. In general, we shall say that some two sets of variables in a system, for example, the state set $\{x_i\}$ and the output set $\{y_j\}$, are non-interacting if and only if the values which the elements of the set $\{y_j\}$ can assume do not depend on the values which the elements of the set $\{x_i\}$ can assume.

Hence, it follows immediately the

Proposition (9.1). Consider the linear dynamical system Eq. (5.2). The set of variables $\{\xi_i\}$, $i \epsilon I$, and the set of variables $\{\eta_j\}$, $j \epsilon J$ where ξ_i and η_j denote the components of either input, state, or output vectors, are non-interacting sets if and only if the set $\{\xi_i\}$, $i \epsilon I$, is invariant with respect to the set $\{\eta_j\}$, $j \epsilon J$. Otherwise the sets $\{\xi_i\}$ and $\{\eta_j\}$ are interacting.

On the basis of the above proposition, we can apply in a straightforward manner the appropriate criteria of invariance to analyze the interaction structure of a given system. To obtain a complete picture of the interaction structure, the analysis would have to cover all possible combinations of the subsets of the sets involved. Fortunately, it is enough to establish the existence or non-existence of interaction between individual elements of the sets. These interaction or non-interaction relations between the individual elements of the sets will be called basic interactions. It will be shown that interactions between any subsets of the sets can be derived from the basic interactions.

It has been shown (see page 94) that, for example, the set of state components $\{x_i\}$, $i \epsilon I$ (where I is some index set), is invariant with respect to some input u_j, $j = 1, 2, \ldots, r$, if and only if each element x_i, $i \epsilon I$, is invariant with respect to u_j.

Furthermore, some component x_i, $i = 1, 2, .., n$, of the state vector $\underline{x}(t)$ is invariant with respect to the input set $\{u_j\}$, $j\varepsilon J$, if and only if x_i is invariant with respect to each element of the set $\{u_j\}$, $j\varepsilon J$, individually.

We can generalize the above statement in terms of the proposition 9.2 as follows:

Proposition (9.2). Consider the linear dynamical system (5.2). The set of variables $\{\xi_i\}$, $i\varepsilon I$, and the set of variables $\{\eta_j\}$, $j\varepsilon J$, where ξ_i and η_j denote components of either input, state, or output vectors, are non-interacting sets if and only if every element of the set $\{\xi_i\}$ is invariant with respect to every element of the set $\{\eta_j\}$. Otherwise the sets $\{\xi_i\}$ and $\{\eta_j\}$ are interacting.

Now, let ρ denote the interaction relation in the sets $\{\xi_i\}$ and $\{\eta_j\}$, and define the matrix (r_{ij}) by

$$r_{ij} = \begin{cases} 1 & \text{if} \quad \xi_i \rho \eta_j \quad \text{for some } i\varepsilon I \text{ and } j\varepsilon J \\ 0 & \text{otherwise} \end{cases}$$

where $\xi_i \rho \eta_j$ means that the element ξ_i of the set $\{\xi_i\}$, $i\varepsilon I$, is interacting with the element η_j of the set $\{\eta_j\}$, $j\varepsilon J$. This matrix shall be called basic interaction matrix. Evidently, the procedure for constructing a basic interaction matrix can be reversed. Given any k x l matrix

$$\begin{bmatrix} r_{11}r_{12} \text{-----} r_{1l} \\ r_{21}r_{22} \text{-----} r_{2l} \\ \text{- - - - - - - - -} \\ \text{- - - - - - - - -} \\ r_{k1}r_{k2} \text{-----} r_{kl} \end{bmatrix}$$

with coefficients $r_{ij} \varepsilon B = \{0, 1\}$, we can define an interaction relation

$$\rho : \Sigma \times \Sigma \rightarrow B$$

on $\Sigma = \{\zeta_1, \ldots., \zeta_l\}$ to $\Sigma = \{\eta_1, \ldots, \eta_k\}$ by setting

$$\rho(\zeta_i, \eta_j) = r_{ij} \quad (1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l)$$

<u>Example 25</u>. Consider the system given by the state equation

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \underline{u}(t)$$

We can easily verify, by applying Theorem $\overline{SC}$(I, 4444) to each pair (u_j, x_i), that the basic input-state interaction matrix is

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

From the basic interaction matrix we can obtain the complete interaction matrix, indicating interaction relationship between any subsets of the sets $\{\zeta_i\}$ and $\{\eta_j\}$, by the following simple procedure.

Let $\{\zeta_i\}$, $i\varepsilon I$, and $\{\eta_j\}$, $j\varepsilon J$ (where I and J are some index sets) be subsets of the sets $\{\zeta\}$ and $\{\eta\}$, representing the components of vectors $\underline{\xi}$ and $\underline{\eta}$. Denote by ρ the interaction relation between the subsets $\{\zeta_i\}$, $i\varepsilon I$, and $\{\eta_j\}$, $j\varepsilon J$, i.e. $\{\zeta_i\} \rho \{\eta_j\}$. The matrix (s_{ij}) defined by

$$s_{ij} = \begin{cases} 1 & \text{if } \zeta_i \rho \eta_j \text{ for some } \xi_i \varepsilon \{\xi_i\} \\ & \text{and } \eta_j \varepsilon \{\eta_j\} \\ 0 & \text{otherwise} \end{cases}$$

represents the complete structure of interaction in the system and is called the <u>interaction matrix</u>.

<u>Example 26</u>. In the previous example, the complete interaction matrix is

$$\begin{array}{c|ccccccc|}
 & \{u_1\} & \{u_2\} & \{u_3\} & \{u_1,u_2\} & \{u_1,u_3\} & \{u_2,u_3\} & \{u_1,u_2,u_1\} \\
\hline
\{x_1\} & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{x_2\} & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
\{x_3\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{x_1,x_2\} & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{x_1,x_3\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{x_2,x_3\} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\{x_1,x_2,x_3\} & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Hence, in the above system, the noninteracting input-state sets are: $\{u_1\} \bar{\rho} \{x_1\}$, $\{u_1\} \bar{\rho} \{x_2\}$, $\{u_2\} \bar{\rho} \{x_2\}$, $\{u_1\} \bar{\rho} \{x_1,x_3\}$ and $\{u_1,u_2\} \bar{\rho} \{x_2\}$.

In the remainder of this chapter we shall be concerned with only two types of interactions, i.e.,

(A) Input-state interactions between sets $\{u_i\}$, $i\varepsilon I$, and $\{x_j\}$, $j\varepsilon J$.

(B) Input-output interactions between sets $\{u_i\}$, $i\varepsilon I$, and $\{y_k\}$, $k\varepsilon K$.

where I, J, K are some index sets. Interactions of this type occur very often in real life systems and it will be shown here that both the analysis and synthesis of interacting systems can be conveniently performed in terms of controllability, observability, and invariance criteria given in the preceding chapters.

When we are dealing with above Type A or Type B interactions a necessary condition that some set $\{\zeta_i\}$, $i\varepsilon I$, be selectively controllable by some set $\{\eta_j\}$, $j\varepsilon J$, is that the sets $\{\zeta_i\}$ and $\{\eta_j\}$ are interacting. If they are not interacting, then by definition of non-interaction $\{\zeta_i\}$ is invariant with respect to $\{\eta_j\}$ and hence cannot be controlled by $\{\eta_j\}$. However, the opposite is in general not true: from the fact that $\{\zeta_j\}$ is not invariant with respect to the set $\{\eta_i\}$ it does not follow that the set $\{\zeta_j\}$ is controllable by the set $\{\eta_i\}$. This follows immediately from the theorems of the class $\overline{SC}$(, 5555) or $\overline{OC}$(, 5555). It is only in the case of a single element of the set $\{\zeta_i\}$ (i.e. a single component of the vect or or $\underline{\zeta}$) interacting with a single component of the set $\{\eta_j\}$ that the presence of interaction implies controllability. With this in mind, we shall define two major categories of interactions as follows:

Definition (9.1). Let $\{u_i\}$, $i\varepsilon I$, be a set of the components of the input vector $\underline{u}(t)$ and $\{x_j\}$, $j\varepsilon J$, be a set of components of the state vector $\underline{x}(t)$ [or $\{y_k\}$, $k\varepsilon K$, a set of the components of the output vector $\underline{y}(t)$]. We shall say that the sets $\{u_i\}$ and $\{x_j\}$ [or $\{y_k\}$] are weakly interacting if and only if the set $\{x_j\}$ [or $\{y_k\}$] is variant * but not controllable by the set $\{u_i\}$. We shall say that the sets $\{u_i\}$ and $\{x_j\}$ [or $\{y_k\}$] are strongly interacting if and only if the set $\{x_j\}$ [or $\{y_k\}$] is controllable by the set $\{u_i\}$.

We shall show that under certain circumstances weak interactions between sets $\{u_i\}$ and $\{x_j\}$ [or $\{y_k\}$] can be eliminated by equivalence transformation of the system under consideration.

9.2. Input-State Interaction. On the basis of preceding

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* Here and in the sequel the term variant will be used to replace the rather clumsy expression "not invariant".

propositions and theorems, one can analyze the input-state interactions of a given system in greater depth, i.e., one might want to specify not only the fact that some sets $\{u_i\}$ and $\{x_j\}$ are interacting, but also whether they are weakly or strongly interacting. With respect to the individual components of the input set $\{u_i\}$ and state set $\{x_j\}$, there is only one alternative as stated in the preceding section on interaction in general: either a $x_i \varepsilon\{x\}$ is non-interacting with respect to some $u_j \varepsilon\{u\}$, or otherwise it is strongly interacting. However, for subsets containing more than one element we can have distinct cases of non-interaction, weak interaction or strong interaction. If the analysis using criteria described in the preceding section shows that some two sets $\{u_i\}$ and $\{x_j\}$ are interacting, it is further necessary to verify controllability in order to determine whether interaction is weak or strong. However, the following propositions simplify this task to some extent.

<u>Proposition (9.3)</u>. The set of state components $\{x_i\}$, $i \varepsilon I$, is weakly interacting with the set of input components $\{u_j\}$, $j \varepsilon J$, if and only if there is at least one element in the set $\{u_j\}$, $j \varepsilon J$, with which the set $\{x_i\}$, $i \varepsilon I$ is weakly interacting, and there is no element in the set $\{u_j\}$, $j \varepsilon J$, with which the set $\{x_i\}$, $i \varepsilon I$, is strongly interacting.

<u>Proposition (9.4)</u>. The set of state components $\{x_i\}$, $i \varepsilon I$, is strongly interacting with the set of input components $\{u_j\}$, $j \varepsilon J$, if there is at least one component in the set $\{u_j\}$, $j \varepsilon J$, with which the set $\{x_i\}$, $i \varepsilon I$, is strongly interacting.

Note that the last proposition gives only sufficient condition for strict interaction, but the condition is not necessary. It is possible to have situations where there might be no strong interaction between a set of state components $\{x_i\}$, $i \varepsilon I$, and the individual elements of the set $\{u_j\}$, $j \varepsilon J$, but the sets $\{x_i\}$ and $\{u_j\}$ as such are nevertheless strongly interacting. Hence, in such cases it is necessary to apply the criteria for strong interaction, given in the definition, directly.

Example 27. Consider the system given by the state equation

$$\underline{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \underline{u}(t)$$

For this system, we have

$$Q_1 = [\underline{b}_{\cdot 1} \vdots A\, \underline{b}_{\cdot 1} \vdots A^2 b_{\cdot 1}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$Q_2 = [\underline{b}_{\cdot 2} \vdots A\, \underline{b}_{\cdot 2} \vdots A^2 \underline{b}_{\cdot 2}] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and

$$Q = [B \vdots A\, B \vdots A^2 B] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Let us denote non-interaction by 0, weak interaction by 1, and strong interaction by ①. After checking invariance and controllability of various sets of state components, with respect to various input combinations, we can summarize the results in the Table below:

	$\{u_1\}$	$\{u_2\}$	$\{u_1, u_2\}$
$\{x_1\}$	①	①	1
$\{x_2\}$	①	0	1
$\{x_3\}$	①	①	1
$\{x_1,x_2\}$	①	①	1
$\{x_1,x_2\}$	①	①	①
$\{x_2,x_3\}$	①	①	1
$\{x_1, x_2,x_3\}$	①	①	①

The first three entries in the column $\{u_1,u_2\}$ have been obtained from the entries in the first two columns by using the Proposition (9.4) but the remaining entries had to be obtained by direct calculation.

9.3. Input-Output Interaction. What has been said so far about interactions in general and about input-state interaction in particular, is also applicable to input-output interactions with just a simple change in wording.

The input-output interaction matrix can be calculated directly by first calculating the basic input-output interaction matrix according to the criteria of output controllability (Proposition 9.2). However, it can be also obtained from the basic input-state interaction matrix by the following:

Theorem (9.1). Consider the system Equation (5.2). Let T_s

denote the basic input-state interaction matrix. Then the basic input-output interaction matrix T_o is given by

$$T_o = \tilde{F} \times T_s + \tilde{G}$$

where $\tilde{F}$ and $\tilde{G}$ are the zero-one matrices derived from the matrices F and G using the following rule

$$(\tilde{f}_{ij}) = \begin{cases} 0 & \text{if} \quad f_{ij} = 0 \\ 1 & \text{otherwise} \end{cases}$$

and

$$(\tilde{g}_{ij}) = \begin{cases} 0 & \text{if} \quad g_{ij} = 0 \\ 1 & \text{otherwise} \end{cases}$$

and the operations x and + represent Boolean matrix multiplication and logical addition, respectively.

Proof. Consider the output equation of the system Equation (5.2) i.e.,

$$\underline{y}(t) = F\,\underline{x}(t) + G\,\underline{u}(t)$$

In order that there were no interaction between some component $u_i(t)$, $i = 1, 2, \ldots, r$, of the input vector $\underline{u}(t)$ and some component $y_j(t)$, $j = 1, 2, \ldots, p$, of the output vector $\underline{y}(t)$ it is obviously necessary that

(1) the component y_j would not interact with the component u_i

via direct transmission (represented by the second term on the right):

(2) the component y_j would not interact with those elements of the set of components $\{x_j\}$ which interact with u_i.

Assume first that $G = 0$. Then there is no interaction between some component y_j and x_k if and only if the element f_{jk} of the matrix F is equal to zero. Otherwise the components y_j and x_k interact. If we construct a new matrix $\tilde{F}$ by letting

$$(\tilde{f}_{ij}) = \begin{cases} 0 & \text{if} \quad f_{ij} = 0 \\ 1 & \text{otherwise} \end{cases}$$

then the matrix $\tilde{F}$ will represent the structure of interaction between the sets of variables $\{y_k\}$ and $\{x_j\}$.

Now, if $u_i \rho x_j$ denotes the interaction relation between u_i and x_j and $x_j \rho y_k$ denote the interaction relation between x_j and y_k, then it is clear that

$$u_i \rho^* y_k = \begin{cases} 1 & u_i \rho x_j = 1 \quad \text{and} \quad x_j \rho y_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

so that the basic input-output interaction matrix T_o, when $G = 0$, can be obtained by taking the Boolean matrix product

$$T_o = \tilde{F} \times T_s$$

If $G \neq 0$, then there is interaction between some elements u_i, $i = 1, 2, \ldots, r$ and y_k, $k = 1, 2, \ldots, p$, because of the direct transmission whenever $y_{ik} \neq 0$. This structure is represented by the matrix $\tilde{G}$ in which

$$(\tilde{g}_{ik}) = \begin{cases} 0 & \text{if} \quad g_{ik} = 0 \\ 1 & \text{otherwise} \end{cases}$$

Since

$$u_i \rho y_k = \begin{cases} 0 & \text{if} \quad u_i \rho^* y_k = 0 \quad \text{and} \quad \tilde{g}_{ik} = 0 \\ 1 & \text{otherwise} \end{cases}$$

we get

$$T_o = \tilde{F} \times T_s + \tilde{G}$$

where x and + represent the operations of Boolean matrix multiplication and logical addition, respectively.

Example 28. Consider the system S given by the equations

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 9 & 0 \\ -1 & 2 & 3 \\ 0 & 0 & -6 \\ 4 & 0 & 4 \end{bmatrix} \underline{u}(t)$$

The basic input-state interaction matrix was found to be (see Example 25).

$$T_s = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

With

$$\tilde{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\tilde{G} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

we find the input-output interaction matrix

$$T_o = \tilde{F} \times T_s + \tilde{G} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{array}{c c} & \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

The interaction matrix shows that there is no interaction between the components y_1 and u_1, y_3 and u_1 and y_3 and u_2. The correctness of these results can be easily verified by direct computation.

The above basic input-output interaction matrix can be extended to a complete interaction matrix by the procedure outlined on page 132. In this instance, we obtain

	$\{u_1\}$	$\{u_2\}$	$\{u_3\}$	$\{u_1,u_2\}$	$\{u_1,u_3\}$	$\{u_2,u_3\}$	$\{u_1,u_2,u_3\}$
$\{y_1\}$	0	1	1	1	1	1	1
$\{y_2\}$	1	1	1	1	1	1	1
$\{y_3\}$	0	0	1	0	1	1	1
$\{y_4\}$	1	1	1	1	1	1	1
$\{y_1,y_2\}$	1	1	1	1	1	1	1
$\{y_1,y_3\}$	0	1	1	1	1	1	1
$\{y_1,y_4\}$	1	1	1	1	1	1	1
$\{y_2,y_3\}$	1	1	1	1	1	1	1
$\{y_2,y_4\}$	1	1	1	1	1	1	1
$\{y_3,y_4\}$	1	1	1	1	1	1	1
$\{y_1,y_2,y_3\}$	1	1	1	1	1	1	1
$\{y_1,y_2,y_4\}$	1	1	1	1	1	1	1
$\{y_1,y_3,y_4\}$							
$\{y_2,y_3,y_4\}$	1	1	1	1	1	1	1
$\{y_1,y_2,y_3,y_4\}$	1	1	1	1	1	1	1

In addition to strong interaction between some input and output sets $\{u_i\}, i\varepsilon I$, and $\{y_k\}$ $k\varepsilon K$, which is analogous in definition with strong input-state interaction, we can identify and define several other types of input and output interaction, depending on whether or not the interacting sets are observable or controllable and observable. Of particular interest is the case of <u>regulatory interaction</u> which we define as follows:

Definition (9.2). Consider some input, state, and output sets $\{u_i\}$, $i \varepsilon I$, $\{x_j\}$, $j \varepsilon J$, and $\{y_k\}$ $k \varepsilon K$. If the set $\{u_i\}$ is strictly interacting with the set $\{x_j\}$ [i.e., if the set $\{x_j\}$ is controllable by the set $\{u_i\}$] and if the set $\{x_j\}$ is observable in the set $\{y_k\}$, then we shall say that the sets $\{u_i\}$ and $\{y_k\}$ are regulatorily interacting sets.

This type of interaction appears in problems where we are concerned with regulating the output and is a sufficient condition for the control to exist [15]. By regulating the output it is understood the capability to:

(1) drive the output to zero;

(2) determine the control which will maintain the output at zero thereafter.

Example 29. Consider the linear constant-parameter dynamical system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \underline{u}(t)$$

for which we have

$$P_1 = [F\,\underline{b}_{.1} \vdots F\,A\,\underline{b}_{.1} \vdots F\,A^2\,\underline{b}_{.1} \vdots \underline{g}_{.1}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P_2 = [F\,\underline{b}_{.2} \vdots F\,A\,\underline{b}_{.2} \vdots F\,A^2\,\underline{b}_{.2} \vdots \underline{g}_{.2}] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

If we denote, as in the previous examples, non-interaction by 0, weak interaction by 1, and strong interaction by 1 , we can represent the complete interaction structure in the system as shown in the table below:

	$\{u_1\}$	$\{u_2\}$	$\{u_1,u_2\}$
$\{y_1\}$	0	1	1
$\{y_2\}$	1	1	1
$\{y_3\}$	1	0	1
$\{y_1,y_2\}$	1	1	1
$\{y_1,y_3\}$	1	1	1
$\{y_2,y_3\}$	1	1	1
$\{y_1,y_2,y_3\}$	1	1	1

CHAPTER X

SYNTHESIS TECHNIQUE

The criteria of interaction between input and state or input and output sets can be readily applied to the design of systems with desired interaction characteristics. The technique has the advantage of giving the system's designer great freedom and versatility in the choice of an interaction structure. He is limited practically just by the physical considerations of feasibility of implementation of the chosen structure. These limitations will be discussed in the next section.

The steps to be followed in the synthesis of a system with given interaction specifications are demonstrated in the following two examples.

Example 30. The objective is to design a linear constant-parameter dynamical system with two inputs and three-dimensional state space, in which the input component $u_1(t)$ would not interact with the set of state components $\{x_1, x_3\}$, and $u_2(t)$ would not interact with $\{x_2\}$.

Since no requirements are placed on the outputs, we shall consider in this example only the state equation

$$\underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$

where A is a 3 x 3 matrix, B is a 3 x 2 matrix, and the elements of A and B are to be determined so that the system would meet the above non-interaction requirements.

First we construct the basic interaction matrix, which in this case is

$$\begin{array}{c|cc} & u_1 & u_2 \\ \hline x_1 & 0 & 1 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{array}$$

Applying the critera for non-interaction given in Proposition 9.1 (p. 150) and Theorem $\overline{\text{SI}}$(I, 4444) for selective state invariance, we find that the system's parameters have to satisfy the following requirements:

a) for $u_1(t)$ to non-interact with $x_1(t)$, we must have

$$\det \begin{bmatrix} b_{11} & -a_{12} & -a_{13} \\ b_{21} & s-a_{22} & -a_{23} \\ b_{31} & -a_{32} & s-a_{33} \end{bmatrix} = 0$$

b) for $u_1(t)$ to non-interact with $x_3(t)$, we must have

$$\det \begin{bmatrix} s-a_{11} & -a_{12} & b_{11} \\ -a_{21} & a-a_{22} & b_{21} \\ -a_{31} & -a_{32} & b_{31} \end{bmatrix} = 0$$

c) for $u_2(t)$ to non-interact with $x_2(t)$, we must have

$$\det \begin{bmatrix} s-a_{11} & b_{12} & -a_{13} \\ -a_{21} & b_{22} & -a_{23} \\ -a_{31} & b_{32} & s-a_{33} \end{bmatrix} = 0$$

From a) we get

$$b_{11} [(s-a_{22})(s-a_{33}) - a_{23}a_{32}] + a_{12} [b_{21}(s-a_{33}) + b_{31}a_{23}]$$

$$+ - a_{13}[-b_{21}a_{32} - b_{31}(s-a_{22})] = 0$$

or

$$b_{11} s^2 + (b_{21}a_{12} + b_{31}a_{13} - b_{11}a_{33} - b_{11}a_{22}) s + b_{11} (a_{22}a_{33} - a_{23}a_{32})$$

$$+ b_{21} (a_{13}a_{32} = a_{12}a_{33}) + b_{31} (a_{12}a_{23} + - a_{13}a_{22}) = 0$$

Similarly, from b) and c) we get

$$b_{31} S^2 + (b_{21}a_{32} - b_{31}a_{22} + b_{11}a_{31} - b_{31}a_{11}) S + b_{31} (a_{11}a_{22} - a_{12}a_{21})$$

$$+ b_{21} (a_{31}a_{12} - a_{32}a_{11}) + b_{11} (a_{21}a_{32} - a_{31}a_{22}) = 0$$

$$b_{22} S^2 + (b_{32}a_{23} - b_{22}a_{33} - b_{22}a_{11} + b_{12}a_{21}) S + b_{22} (a_{11}a_{33} - a_{13}a_{31})$$

$$+ b_{32}(a_{13}a_{21} - a_{11}a_{23}) + b_{12}(a_{23}a_{31} - a_{21}a_{33}) = 0$$

For the above three equations to be identically zero, the parameters have to be so chosen as to satisfy the system of equations.

$$\begin{cases} b_{11} = 0 \\ b_{31} = 0 \\ b_{22} = 0 \\ b_{21}a_{12} = 0 \\ b_{21}a_{32} = 0 \\ b_{32}a_{23} + b_{12}a_{21} = 0 \\ b_{21}\ (a_{13}a_{32} - a_{12}a_{33}) = 0 \\ b_{21}\ (a_{31}a_{12} - a_{32}a_{11}) = 0 \\ b_{32}\ (a_{13}a_{21} - a_{11}a_{23}) + b_{12}\ (a_{23}a_{31} - a_{21}a_{33}) = 0 \end{cases}$$

We have now several choices how to solve the above system of equations up to arbitrary constants.

(i) We can let $(b_{ij}) = 0$ for all i and all j which results in $B = 0$. This is, of course, trivial.

(ii) We shall assume that $b_{21} \neq 0$; $b_{32} \neq 0$; $b_{12} \neq 0$. Then it follows that (in most general case)

$$\begin{cases} a_{12} = 0 \\ a_{32} = 0 \\ b_{32}a_{23} = b_{12}a_{21} = 0 \\ b_{32}\ (a_{13}a_{21} - a_{11}a_{23}) + b_{12}\ (a_{23}a_{31} - a_{21}a_{33}) = 0 \end{cases}$$

From the third equation we get

$$b_{32} = - b_{12} \frac{a_{21}}{a_{23}}$$

and substituting into the fourth equation

$$b_{12} \left(- a_{21}^2 \frac{a_{13}}{a_{23}} + \frac{a_{11}a_{21}a_{23}}{a_{23}} + a_{23}a_{31} - a_{21}a_{33} \right) = 0$$

Since we have assumed $b_{12} \neq 0$, we get

$$\frac{a_{13}}{a_{23}} a_{21}^2 + (a_{33} - a_{11})\, a_{21} - a_{23}a_{31} = 0$$

from which equation we can express the element a_{21} in terms of other elements of the matrix A, i.e.

$$a_{21} = \frac{a_{11} - a_{33} \pm \sqrt{(a_{33} - a_{11})^2 + 4\, a_{13}a_{31}}}{2\, a_{13}/a_{23}}$$

Since the parameters are to be real, we are interested only in cases where

$$(a_{33} - a_{11})^2 + 4\, a_{13}a_{31} \geqslant 0$$

If we denote by (a^*_{ij}) and (b^*_{ij}) the element of the matrix A viz. matrix B, the values of which depend on the arbitrary selected values of other elements of these matrices, then the required system solution can be represented by the equation:

$$\underline{\dot{x}}(t) = \begin{bmatrix} a^*_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a^*_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \\ 0 & b_{32} \end{bmatrix} \underline{u}(t)$$

or by the equation

$$\underline{\dot{x}}(t) = \begin{bmatrix} a_{11} & 0 & a^*_{13} \\ a_{21} & a_{22} & a^*_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \\ 0 & b_{32} \end{bmatrix} \underline{u}(t)$$

or by any other system which satisfies the above given relationships of its parameters.

For instance, let

$$b_{12} = 1$$

$$b_{21} = 3$$

$$b_{32} = -1$$

$$a_{21} = 2 \qquad a_{13} = 5 \qquad a_{22} = 1$$

$$a_{31} = 3 \qquad a_{33} = -1$$

Then

$$a_{23} = 2$$
$$a_{11} = 1$$

and we have the system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 2 \\ 3 & 0 & -1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 0 & -1 \end{bmatrix} \underline{u}(t)$$

in which $u_1(t)$ does not interact with $\{x_1, x_3\}$ and $u_2(t)$ does not interact with $\{x_2\}$. However, in this system the input set $\{u_1,u_2\}$ is not strongly interacting with the state set $\{x_1,x_2,x_3\}$, i.e., the system is not state controllable. If this is required, then additional constraints have to be imposed on the system's parameters as shown below.

From Theorem $\overline{SC}$(II, 6666), the necessary and sufficient condition for a system to be state controllable is that the composite matrix

$$Q = [B \vdots A B \vdots A^2B]$$

be of rank 3. For the first alternative system solution we have:

$$P = \begin{bmatrix} 0 & b_{12} & 0 & a_{11}^* b_{12} + a_{13} b_{32} & 0 \\ b_{21} & 0 & a_{22} b_{21} & a_{21} b_{12} + a_{23}^* b_{32} & a_{22}^2 b_{21} \\ 0 & b_{32} & 0 & a_{31} b_{12} + a_{33} b_{32} & 0 \end{bmatrix}$$

$$\left.\begin{matrix}
(a_{11}^{*2} + a_{13}\, a_{31})\, b_{12} \qquad\qquad + (a_{11}^{*}\, a_{13} + a_{13}\, a_{33})\, b_{32} \\
(a_{21}\, a_{11}^{*} + a_{22}\, a_{21} + a_{23}^{*}\, a_{31})\, b_{12} + (a_{21}\, a_{13} + a_{22}\, a_{23}^{*} + a_{23}^{*}\, a_{33})\, b_{32} \\
(a_{31}\, a_{11}^{*} + a_{33}\, a_{31})\, b_{12} + (a_{31}\, a_{13} + a_{33}^{2})\, b_{32}
\end{matrix}\right]$$

For P to be of rank 3, we can require, for instance that

$$\det \begin{bmatrix} 0 & b_{12} & a_{11}^{*}\, b_{12} + a_{13}\, b_{32} \\ b_{21} & 0 & a_{21}\, b_{12} + a_{23}^{*}\, b_{32} \\ 0 & b_{32} & a_{31}\, b_{12} + a_{33}\, b_{32} \end{bmatrix} \neq 0$$

which is equivalent to the requirement that

$$b_{12}\, (a_{31}\, b_{12} + a_{33}\, b_{32}) \neq b_{32}\, (a_{11}^{*}\, b_{12} + a_{13}\, b_{32})$$

This as well as the preceding non-interaction requirements are satisfied, for example, by the following choice of parameters:

$$\begin{matrix} b_{12} = 1 & ; & b_{21} = 3 & ; & b_{32} = -1 \\ a_{21} = 2 & & a_{22} = 1 & & a_{13} = 5 \\ a_{31} = 0 & & a_{33} = -1 & & \end{matrix}$$

Then

$$a_{23}^{*} = 2$$

$$a_{11}^{*} = 4$$

and the system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 4 & 0 & 5 \\ 2 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 0 & -1 \end{bmatrix} \underline{u}(t)$$

is non-interacting in the sets $\{u_1\}$ and $\{x_1, x_3\}$, and $\{u_2\}$ and $\{x_2\}$, but strongly interacting (i.e., controllable) with respect to the sets $\{u_1, u_2\}$ and $\{x_1, x_2, x_2\}$.

<u>Example 31</u>. Let us assume that we are required to design a single input, three output linear dynamical system

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + \underline{b}\,u(t)$$

$$\underline{y}(t) = F\,\underline{x}(t) + \underline{g}\,u(t)$$

in which the input would not interact with the first component $y_1(t)$ of the output vector $\underline{y}(t)$. Here we assume $\underline{x}(t)$ a 3-component vector, A a 3 x 3 matrix, $\underline{b}$ and $\underline{g}$ 3-component constant vectors, and F a 3 x 3 matrix. Of course, we do not have to limit ourselves to $\underline{x}(t)$ being a 3-component vector; other choices are possible and they would usually depend on real life requirements in the selection of a proper state variable.

The basic input-output interaction matrix for this system is

	u
y_1	0
y_2	1
y_3	1

The basic input-state interaction matrix is in this case not unique and any choice is acceptable (in absence of further design specifications) as long as the requirement

$$T_o = \tilde{F} \times T_s + \tilde{G}$$

stated in Theorem 9.1 is satisfied. Hence, we have to choose the interaction matrices $\tilde{F}$, $\tilde{G}$ and T_s such that

$$\tilde{F} \times T_s + \tilde{G} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For instance, one of the possible choices is

$$\tilde{G} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$T_s = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{F} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The choice of T_s indicates that we require that the input $u(t)$ would not interact with the second component $x_2(t)$ of the state vector $\underline{x}(t)$.

To satisfy this requirement, it is necessary that

$$\det \begin{bmatrix} s-a_{11} & b_1 & -a_{13} \\ -a_{21} & b_2 & -a_{23} \\ -a_{31} & b_3 & s-a_{33} \end{bmatrix} = 0$$

or

$$b_2 S^2 + (a_{23} b_3 - a_{33} b_2 - a_{11} b_2 + a_{21} b_1) S + (a_{23} a_{31} - a_{21} a_{33}) b_1$$

$$+ (a_{11} a_{33} - a_{13} a_{31}) b_2 + (a_{13} a_{21} - a_{11} a_{23}) b_3 = 0$$

Hence

$$b_2 = 0$$

$$a_{2}, b_1 + a_{23} b_3 = 0$$

$$(a_{23} a_{31} - a_{21} a_{33}) b_1 + (a_{13} a_{21} - a_{11} a_{23}) b_3 = 0$$

If we require that $b_1 \neq 0$ and $b_3 \neq 0$, then four of the a's can be chosen arbitrary (within certain limits). Again let a^*_{ij} denote the elements which depend on our choice of other a's. For example, we might select the system

$$\underline{\dot{x}}(t) = \begin{bmatrix} a_{11}^{*} & a_{12} & a_{13} \\ a_{21}^{*} & a_{22} & a_{23} \\ a_{21}^{*} & a_{22} & a_{23} \end{bmatrix} \underline{x}(t) + \begin{bmatrix} b_1 \\ 0 \\ b_3 \end{bmatrix} u(t)$$

From F and G, we can write the most general output equation

$$\underline{y}(t) = \begin{bmatrix} 0 & f_{12} & 0 \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ g_2 \\ g_3 \end{bmatrix} u(t)$$

and the system thus obtained satisfies the non-interaction requirements specified at the beginning.

To give a numerical example, let us choose the following values for the "free" parameters of the state equation

$$\begin{array}{lll} b_1 = 2 & & b_3 = 2 \\ & a_{12} = 2 & a_{13} = 2 \\ & a_{22} = 2 & a_{23} = 2 \\ a_{31} = 2 & a_{32} = 2 & a_{33} = 2 \end{array}$$

Then

$$a_{21}^{*} = -2$$

$$a_{11}^{*} = 2$$

and

$$\underline{\dot{x}}(t) = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} u(t)$$

We can easily verify that, independent of the choice of non-zero elements in the matrices F and $\underline{g}$ of the output equation, the system is $y_1(t)$ output invariant. In fact

$$H = [F\,\underline{b} \,\vdots\, F\,A\,\underline{b} \,\vdots\, F\,A^2\underline{b}]$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 2f_{21} + 2f_{23} & 8f_{21} + 8f_{23} & 32f_{21} + 32f_{23} \\ 2f_{31} + 2f_{33} & 8f_{31} + 8f_{33} & 32f_{31} + 32f_{33} \end{bmatrix}$$

and by Theorem $\overline{\text{OI}}$(I, 4444) the necessary and sufficient conditions for above invariance are satisfied.

Let us now assume that there is additional requirement that the system be strongly interacting with respect to input $u(t)$ and the output set $\{y_2, y_3\}$, i.e., be controllable in the subspace of the output space generated by $[0, y_2, 0]^T$ and $[0, 0, y_3]^T$ vectors. By Theorem $\overline{\text{OC}}$(I, 5555), additional restraints are imposed on the systems parameters by the requirement that the matrix

$$\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\left[\begin{matrix} 0 & (a^*_{21}\, b_1 + a_{23}b_3)f_{12} \\ f_{21}b_1+f_{23}b_3 & (a^*_{11}\, b_1 + a_{13}b_3)f_{21} + (a^*_{21}b_1 + a_{23}b_3)f_{22} \\ f_{31}b_1+f_{33}b_3 & (a^*_{11}\, b_1+a_{13}b_3)f_{31} + (a^*_{21}b_1+a_{23}b_3)f_{22}\end{matrix}\right.$$

$$\begin{matrix} & (b_1\, \Sigma\, a_{1j}a_{j1} + b_3\, \Sigma\, a_{1j}a_{j3})f_{21} \\ +\ (a_{31}b_1+a_{33}b_3)f_{23} & (b_1\, \Sigma\, a_{1j}a_{j1} + b_3\, \Sigma\, a_{1j}a_{j3})f_{31} \\ +\ (a_{31}b_1+a_{33}b_3)f_{33} & (b_1\, \Sigma\, a_{2j}a_{j1}+b_3\, \Sigma\, a_{23}a_{j3})f_{12}\end{matrix}$$

$$\left.\begin{matrix} & 0\\ +\ (b_1\, \Sigma\, a_{2j}a_{j1}+b_3\, \Sigma\, a_{2j}a_{j3})f_{22} + (b_1\, \Sigma\, a_{3j}a_{j1}+b_3\, \Sigma\, a_{3j}a_{j3})f_{23} & g_2 \\ +\ (b_1\, \Sigma\, a_{2j}a_{j1} + b_3\, \Sigma\, a_{2j}a_{j3})f_{32} + (b_1\, \Sigma\, a_{3j}a_{j1}+b_3\, \Sigma\, a_{3j}a_{j3})f_{33} & g_3\end{matrix}\right]$$

be of rank 2. We can satisfy this requirement, for instance, by choosing the parameters so that

$$\det \begin{bmatrix} f_{21}b_1 + f_{23}b_3 & g_2 \\ f_{31}b_1 + f_{33}b_3 & g_3 \end{bmatrix} \neq 0$$

or

$$(f_{21}b_1 + f_{23}b_3)\; g_3 \;\neq\; (f_{31}b_1 + f_{33}b_3)\; g_2$$

With the previous numerical choice of elements of the matrix B, we must have

$$(f_{21} + f_{23})\, g_3 \neq (f_{31} + f_{33})\, g_2$$

viz.

$$\frac{g_3}{g_2} \neq \frac{f_{31} + f_{33}}{f_{21} + f_{23}}$$

As long as this inequality is satisfied, any choice of g_2, g_3, f_{31}, f_{33}, f_{21}, and f_{23} will meet the interaction requirements. For instance, we could choose

$$f_{12} = f_{21} = f_{22} = f_{23} = f_{31} = f_{32} = f_{33} = 1$$

$$g_2 = 2 \quad \vdots \quad g_3 = 1$$

and thus obtain the system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$\underline{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} u(t)$$

which meets all the specified interaction requirements.

We shall conclude this section with a theorem which can serve as a tool for eliminating certain undesirable interactions by appropriate transformations of system's variables.

Theorem (10.1). Consider the system Equation (5.2) with the state space Σ and output space R^p. Let the sets $\{u_i\}$ and $\{x_j\}$ or $\{y_k\}$ be weakly interacting, and let the dimension of $\{x_i\}$ be equal to m [or that of $\{y_k\}$ be equal to q]. Moreover, let $Q(u)$ [$P(u)$] be the state [output] controllability matrix of rank i [rank j], $i < m$ [$j < q$], corresponding to $\{u_j\}$. If $m \leq n - i$ [$q \leq p - j$], then there exists an equivalent system S^* in which the sets $\{u_i\}$ and $\{x_j\}$ [or $\{y_k\}$] are non-interacting.

Proof. First let us consider the case of weak interaction between sets $\{u_i\}$ and $\{x_j\}$, where $\{x_j\}$ is of dimension m. If the rank of the controllability matrix Q corresponding to the set $\{u_i\}$ is i, then by Theorem 7.1 there exists an equivalence transformation which transforms the system with the input set $\{u_i\}$ into a system S^* which is invariant in n-i coordinates of the state space Σ. Hence, if the rank of $\{x_j\}$ is $m \leq n - i$ then by properly choosing the basis of equivalence transformation, we can find a system equivalent to the system S^* in which the sets $\{u_i\}$ and $\{x_j\}$ are non-interacting.

CHAPTER XI

PHYSICAL REALIZABILITY OF INTERACTION STRUCTURES

There are certain limitations with regard to the choice of interaction structures, i.e., some design specifications for non-interaction might not make sense for physical systems at all or they might not be realizable from the viewpoint of hardware requirements. Some of the most important limitations are briefly discussed in this section. All the statements below refer to the system Equation (5.2).

(1) It is impossible to construct a state (or output) invariant system in which disturbance input would be applied directly to the state (or output) component which is to remain invariant under that disturbance.

In other words, we are saying that it is impossible to construct a system in which, say, the state component $x_i(t)$, $i = 1,2, \ldots, n$ [or output component $y_k(t)$, $k = 1,2,\ldots, p$] would be invariant with respect to the input component $u_j(t)$, $j = 1,2, \ldots, r$ with $b_{ij} \neq 0$, [or $g_{ij} \neq 0$]. Hence the necessary, but not sufficient, condition that the state component $x_i(t)$ [or output component $y_k(t)$] be invariant with respect to the input (disturbance) component $u_j(t)$ is that the b_{ij}-th element of the matrix B [or the g_{ij}-th element of the matrix G] be equal to zero.

We shall prove this statement first for the case of state invariance, then for the case of output invariance.

By Theorem $\overline{\mathrm{SI}}$(I, 4444), the necessary and sufficient condition for the system (5.2) to be selectively uniformly i-th state invariant with respect to the j-th component $u_j(t)$, $j = 1,2, \ldots, r$ of the input (disturbance) vector $\underline{u}(t)$, is that

$$\det {}^{i}\Gamma^{j}(s) = 0$$

We recall that (assuming zero initial conditions)

$$\det {}^{i}\Gamma^{j}(s) = \begin{bmatrix} s-a_{11} & -a_{12} \ldots b_{ij} & \cdot\ \cdot\ \cdot -a_{1n} \\ -a_{21} & s-a_{22} \ldots b_{2j} & \cdot\ \cdot\ \cdot -a_{2n} \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ -a_{n1} & -a_{n2} \ldots b_{nj} & \cdot\ \cdot\ s-a_{nn} \end{bmatrix} = 0$$

The above determinant can be written in the polynomial form

$$\det {}^{i}\Gamma^{j}(s) = b_{ij}\, s^{n-1} + c_{n-2}\, s^{n-2}$$

$$+ c_{n-3}\, s^{n-3} + \ldots + c\, s + c_o = 0$$

where the coefficients $c_{n-2}, c_{n-3} \ldots, c_o$ are certain products of the elements a's and b's contained in the above determinant, and b_{ij} is the coefficient of the leading term. It is clear now that for the $\det {}^{i}\Gamma^{j}(s)$ to be identically zero, it is necessary that $b_{ij} = 0$. Of course, the coefficients $c_{n-2}, \ldots, c_o$ have to be zeroes as well.

The proof for the case of output invariance follows immediately from Theorem $\overline{OI}$(I, 4444).

(2) It is impossible to construct a system (except in the trivial case of zero input) in which all the components of the state vector (or output vector) were simultaneously invariant with respect to some input component $u_j(t)$.

The proof follows immediately from the proof in the preceding paragraph (1). Namely, from the requirement that all $x_i(t)$, $i = 1,2, \ldots, n$ be invariant with respect to some input (disturbance) component $u_j(t)$, $j = 1,2, \ldots, r$, it follows that $b_{ij} = 0$ for all $i = 1,2, \ldots, n$. But this is the trivial case of zero input along the j-th input component.

For the output-invariance, the above requirement leads to the trivial case of $g_{ij} = 0$ for all $i = 1,2, \ldots, n$.

CHAPTER XII

INTERACTION IN TIME-VARYING SYSTEMS

The procedure of analysis and synthesis of interactions which has been developed for linear time-invariying systems, can be extended to time-varying systems as well. The basic definitions and criteria of interaction and non-interaction remain the same, but the necessary conditions (and sufficiency conditions if they are known) are, of course, different. Furthermore, since there are many more different types of controllability, observability, and invariance for time-varying systems than for linear time-invarying systems for which many of the subtle differences disappear, the types of interactions which can be identified and defined using the proposed methodology are also numerous.

It would go beyond the scope of this work to investigate in detail all the possible interaction situations in linear time-varying systems. At the same time it is not likely that such an investigation would lead to any fundamentally new issues. Therefore we shall briefly discuss only few characteristic aspects of the extension of the proposed methodology to these more complex systems.

Let us first consider linear time-varying system

$$\begin{aligned} \dot{\underline{x}}(t) &= A(t)\,\underline{x}(t) + B(t)\,\underline{u}(t) \\ \underline{y}(t) &= F(t)\,\underline{x}(t) + G(t)\,\underline{u}(t) \end{aligned} \tag{12.14}$$

The essential theorem for the interaction analysis and synthesis is

Theorem SI(I, 111). The i-th state component $x_i(t)$, $i = 1,2, \ldots, n$ of the system (12.14) is selectively invariant with respect to the input component $u_j(t)$, $j = 1,2, \ldots, r$ for all $[t_o, t_1] \subset T$ if and only if the i-th row of the matrix

$$V_j(t) = [\underline{b}_{\cdot j}(t) \vdots Q\underline{b}_{\cdot j}(t) \vdots Q^2\underline{b}_{\cdot j}(t)^1 \ldots \vdots Q^{n-1}\,\underline{b}_{\cdot j}(t)]$$

has only zero elements for all $t \varepsilon [t_o, t_1]$. In the above matrix Q denotes the operator

$$Q = I\frac{d}{dt} - A(t)$$

and $\underline{b}_{\cdot j}(t)$ is the j-th column vector of the matrix $B(t)$. For example $Q\,\underline{b}_{\cdot j}(t) = \underline{b}_{\cdot j}(t) - A(t)\underline{b}_{\cdot j}(t)$.

Similarly, for the output-invariance we have

Theorem OI(I, 111). The i-th output component $y_k(t)$, $k = 1,2, \ldots, p$ of the system (12.14) is selectively invariant with respect to the input component $u_j(t)$, $j = 1,2, \ldots, r$, for all $[t_o, t_1] \subset T$ if and only if the i-th row of the matrix

$$W_j(t) = [F(t)\underline{b}_{\cdot j}(t) \vdots F(t)\,Q\underline{b}_{\cdot j}(t) \vdots \ldots. \vdots F(t)\,Q^{n-1}\,\underline{b}_{\cdot j}(t) \vdots \underline{g}_{\cdot j}(t)]$$

has only zero elements for all $t \varepsilon [t_o, t_1]$. In the above matrix Q is the operator defined in the preceding Theorem SI(I, 111), $\underline{b}_{\cdot j}(t)$ is the j-th column of the matrix $B(t)$, and $\underline{g}_{\cdot j}(t)$ is the j-th column of the matrix $G(t)$.

The proofs of the essential parts of the above theorem can be found in Rozonoer's papers on invariance theory [233, 234] and, even though the results have not been presented in this form, the modifications which are necessary to obtain them are simple enough so that it is not deemed necessary to repeat these proofs here.

The above theorems of invariance can be extended in the manner identical to that for linear time-invariant systems for the various types of selective, complete, strong, or total invariance conditions, or invariance in some subspace of the state or output space.

Since Theorem 7.1 holds for time-varying case as well, the relation between invariance and controllability is immediately established. For instance, the conditions for complete state controllability are given by the

<u>Theorem SC(II, 66)</u>. Consider the system Equation (12.14). The system is completely state controllable if and only if the matrix

$$V(t) = [B(t) \vdots Q\, B(t) \vdots Q^2\, B(t) \vdots \dots \vdots Q^{n-1}\, B(t)]$$

is of rank n for all $t \in T$

<u>Example 32</u>. Consider the system with time-varying co-efficients given by the state equation

$$\underline{\dot{x}}(t) = \begin{bmatrix} -(1 + e^{-t}) & -1 & 0 \\ (1 + 3e^{-t}) & 0 & -1 \\ -3e^{-t} & 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ -1 \\ e^{-t} \end{bmatrix} u(t)$$

Since

$$Q\,\underline{b}(t) = \begin{bmatrix} 0 \\ 0 \\ -e^{-t} \end{bmatrix} - \begin{bmatrix} -(1 + e^{-t}) & -1 & 0 \\ (1 + 3e^{-t}) & 0 & -1 \\ -3e^{-t}) & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ e^{-t} \\ -e^{-t} \end{bmatrix}$$

and

$$Q^2\, b(t) = \begin{bmatrix} 0 \\ -e^{-t} \\ e^{-t} \end{bmatrix} - \begin{bmatrix} (1 + e^{-t}) & 1 & 0 \\ -(1 + 3e^{-t}) & 0 & 1 \\ 3e^{-t} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 + e^{-t} \\ -2e^{-t} \end{bmatrix}$$

we get

$$V(t) = \begin{bmatrix} 0 & -1 & -1 \\ -1 & e^{-t} & 1 + e^{-t} \\ e^{-t} & -e^{-t} & -2e^{-t} \end{bmatrix}$$

Since det $V(t) = 0$ for all $t \in [0, \infty]$, by Theorem SC(II, 66) the system is not completely controllable.

As it has been mentioned, the interaction analysis and synthesis can be handled identically as in the case of linear time-invariant systems. This is illustrated by the following examples.

Example 33. It is required to design a linear time-variant dynamical system with two inputs, two-component state vector $\underline{x}(t)$, and two outputs

$$\underline{\dot{x}}(t) = A(t)\ \underline{x}(t) + B(t)\ \underline{u}(t)$$

$$y(t) = F(t)\ \underline{x}(t)$$

in which the first input component would not interact with the second state component, and the second input would not interact with the first state component. Hence, the basic input state interaction matrix is

$$\begin{array}{c|cc} & x_1 & x_2 \\ \hline u_1 & 1 & 0 \\ u_2 & 0 & 1 \end{array}$$

By Theorem SI(I, 1111), the first condition is satisfied when the second row of the matrix

$$V_1(t) = \begin{bmatrix} b_{11}(t) & b_{11}(t) - a_{11}(t)\ b_{11}(t) - a_{12}(t)\ b_{21}(t) \\ b_{21}(t) & b_{21}(t) - a_{21}(t)\ b_{11}(t) - a_{22}(t)\ b_{21}(t) \end{bmatrix}$$

consists of zero elements, and when the same is true for the first row of the matrix

$$V_2(t) = \begin{bmatrix} b_{12}(t) & b_{12}(t) - a_{11}(t)\ b_{12}(t) - a_{12}(t)\ b_{22}(t) \\ b_{22}(t) & b_{22}(t) - a_{21}(t)\ b_{12}(t) - a_{22}(t)\ b_{22}(t) \end{bmatrix}$$

Hence we have the requirements

$$\begin{cases} b_{21}(t) = 0 \\ \dot{b}_{21}(t) - a_{21}(t)\, b_{11}(t) - a_{22}(t)\, b_{21}(t) = 0 \\ b_{12}(t) = 0 \\ \dot{b}_{12}(t) - a_{11}(t)\, b_{12}(t) - a_{12}(t)\, b_{22}(t) = 0 \end{cases}$$

These conditions are satisfied when

$$b_{12}(t) = 0, \qquad b_{21}(t) = 0$$

$$a_{12}(t) = 0, \qquad a_{21}(t) = 0$$

with the remaining elements of the matrices $A(t)$ and $B(t)$ being arbitrary functions (arbitrary with respect to the problem on hand, but not arbitrary with respect to other system specifications, like stability, sensitivity, etc.) The system is then of the form

$$\underline{\dot{x}}(t) = \begin{bmatrix} a_{11}(t) & 0 \\ 0 & a_{22}(t) \end{bmatrix} \underline{x}(t) + \begin{bmatrix} b_{11}(t) & 0 \\ 0 & b_{22}(t) \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = F(t)\, \underline{x}(t)$$

As one could have expected, the matrices $A(t)$ and $B(t)$ of the system equations are diagonal matrices.

CHAPTER XIII

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

The objective of the research described in this dissertation was to investigate the relation between controllability and invariance of states and outputs of a system and to apply the criteria of controllability and invariance to the design of systems of given interaction specifications. In order not to obscure the main course of this presentation by excessive detail, the investigation was essentially limited to linear time-invariant dynamical systems, but the extension to time variant systems is, at least conceptually, a straightforward procedure. The feasibility of such an extension is illustrated by an example. Furthermore, only continuous time systems were considered, but the transition to discrete time systems is even less complicated.

The relation between state controllability and observability has been established by Kalman in his well known duality theorem, but little was done to clarify the relation between controllability and invariance as well as between observability and invariance. It is shown in this work that such relationships exist, and the derivation of these relationships is presented as part of the overall research results.

The proposed approach of analysis and synthesis of interacting systems was based on the concepts of selective controllability, selective observability, and selective invariance. These concepts were defined within the framework of more general types of controllability, observability, and invariance. Since certain types of controllability, observability, and invariance imply and/or are implied by the other, structures of these implicational relationships were developed and described. Necessary and sufficient criteria for elective controllability, observability, and invariance were derived. For the rest, such criteria were derived only if they were not readily available in subject literature.

The major result of the investigation was the development of a methodology for (1) analysis of interacting dynamical systems and (2) design of dynamical systems with specified interaction or non-interaction characteristics. The analysis and design method which has been developed is based on the systematic application of the criteria of controllability and invariance to subsets of input, state, or output variables, and the flexibility and power of the proposed approach is demonstrated by a number of examples.

Natural extension of this research would be the investigation of the types of interactions, such as output - output interactions, which have not been considered here. Further areas of interest are interactions in non-linear systems and interactions in stochastic systems, of which practically nothing is known.

Complete non-interaction of certain variables of a system is, in the most instances of practical interest, a theoretical abstraction. Normally the best that one can hope to accomplish is to reduce the interactions to such a degree that they could be considered negligible with respect to the design objectives. Problems related to such interactions, which may be referred to as ε-interactions, could be studied by extending the proposed methodology to include criteria of ε-controllability and ε-invariance. For certain types of problems, such criteria are already known.

Finally, it should be noted that in the reported research the emphasis was on structural aspects of the interaction problem in systems. Besides these there are certain other aspects, such as intensity of interaction, which would be desirable to consider in the extension of this work.

APPENDIX 1

STRUCTURAL LISTING OF STATE CONTROLLABILITY CONCEPTS

Input	$u_{j[t_o,\ t_1]}$, $\exists j = 1,2, \ldots, r$			
Controlled Variable \ Time	$\exists t_o,\ t_1 \varepsilon T$	$\forall t_o \varepsilon T, \exists t_1 \varepsilon T$	$\forall t_o,\ t_1 \varepsilon T$	$\forall (t,\tau) \varepsilon TxT$
$\exists\, x_{io}\ \varepsilon\ \Sigma_i$ $i = 1,2, \ldots, n$	SC(I, 1) Selectively controllable i-th state component at t_o	SC(I, 11) Selectively controllable i-th state component	SC(I, 111) Selectively controllable i-th state component for all $[t_o,\ t_1]\ c\ T$	SC(I, 1111) Selectively uniformly controllable i-th state component
$\exists\, \tilde{x}_o\ \varepsilon\ \tilde{\Sigma}$ $\tilde{\Sigma} c \Sigma$	SC(I, 2) Selectively controllable state in $\Sigma c \Sigma$ at t_o	SC(I, 22) Selectively controllable state in Σ	SC(I, 222) Selectively controllable state in Σ , for all $[t_o,\ t_1]\ c\ T$,	SC(I, 2222) Selectively uniformly controllable state in $\tilde{\Sigma}$
$\exists\, \underline{x}_o\ \varepsilon\ \Sigma$	SC(I, 3) Selectively controllable state at t_o	SC(I, 33) Selectively controllable state	SC(I, 333) Selectively controllable state for all $[t_o,\ t_1]\ c\ T$	SC(I, 3333) Selectively uniformly controllable state

APPENDIX I

(continued)

Input	$u_{j[t_o,\ t_1]}, \exists j = 1,2,\ \ldots,r$			
Controlled Variable \ Time	$\exists\ t_o,\ t_1 \varepsilon T$	$\forall\ T_o \varepsilon T, \exists\ t_1 \varepsilon T$	$\forall\ t_o,\ t_1 \varepsilon T$	$\forall\ (t,\tau)\varepsilon Txt$
$\forall x_{io}\ \varepsilon\ \Sigma_i$ $i = 1,2,\ \ldots,n$	SC(I, 4) Selectively i-th state controllable system at t_o	SC(I, 44) Selectively completely i-th state controllable system	SC(I, 444) Selectively totally i-th state-controllable system	SC(I, 4444) Selectively uniformly i-th state controllable system
$\forall \underline{\tilde{x}}_o\ \varepsilon\ \tilde{\Sigma}$ $\tilde{\Sigma}\ c\ \Sigma$	SC(I, 5) Selectively state-controllable system in $\tilde{\Sigma}$ at t_o	SC(I, 55) Selectively completely state controllable system in $\tilde{\Sigma}$	SC(I, 555) Selectively totally state-controllable system in $\tilde{\Sigma}$	SC(I, 5555) Selectively uniformly state-controllable system in $\tilde{\Sigma}$
$\forall \underline{x}_o\ \varepsilon\ \Sigma$	SC(I, 6) Selectively state-controllable system at t_o	SC(I, 66) Selectively completely state controllable system	SC(I, 666) Selectively totally state controllable system	SC(I, 6666) Selectively uniformly state-controllable system

APPENDIX I

(Continued)

Input	$\underline{u}[t_o, t_1]$			
Time / Controlled variable	$\exists\, t_o, t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t_1 \varepsilon T$	$\forall\, t_o, t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon T x T$
$\exists\, x_{io} \in \Sigma_i$ $i = 1,2, \ldots, n$	SC(II, 1) Controllable i-th state component at t_o	SC(II, 11) Controllable i-th state component	SC(II, 111) Selectively controllable i-th state component for all $[t_o, t_1] \subset T$	SC(II, 1111) Proper i-th state component
$\exists\, \underline{\tilde{x}}_o \in \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$	SC(II, 2) Controllable state in $\tilde{\Sigma}$ at t_o	SC(II, 22) Controllable state in $\tilde{\Sigma}$	SC(II, 222) Selectively controllable state in for all $[t_o, t_1] \subset T$	SC(II, 2222) Proper state in $\tilde{\Sigma}$
$\exists\, \underline{x}_o \in \Sigma$	SC(II, 3) Controllable state at t_o	SC(II, 33) Controllable state	SC(II, 333) Controllable state for all $[t_o, t_1] \subset T$	SC(II, 3333) Proper state

APPENDIX I

(Continued)

Input	$\underline{u}[t_o, t_1]$			
Time / Controlled variable	$\exists t_o,\ t_1 \epsilon T$	$\forall t_o \epsilon T,\ \exists t, \epsilon T$	$\forall t_o,\ t_1 \epsilon T$	$\forall (t,\tau) \epsilon T x T$
$\forall x_{io}\ \epsilon\ \Sigma_i$ $i = 1,2, \ldots, n$	SC(II, 4) i-th state controllable system at t_o	SC(II, 44) Completely i-th state controllable system	SC(II, 444) Selectively totally i-th state-controllable system	SC(II, 4444) i-th state-proper system
$\forall \underline{\tilde{x}}_o\ \epsilon\ \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$	SC(II, 5) State controllable system in $\tilde{\Sigma}$ at t_o	SC(II, 55) Completely state controllable system in $\tilde{\Sigma}$	SC(II, 555) Selectively totally state controllable system in $\tilde{\Sigma}$	SC(II, 5555) State proper system in $\tilde{\Sigma}$
$\forall \underline{x}_o\ \epsilon\ \Sigma$	SC(II, 6) State controllable system at t_o	SC(II, 66) Completely state controllable system	SC(II, 666) Totally state controllable system	SC(II, 6666) State-proper system

APPENDIX I

(Continued)

Input	$u_j[t_o, t_1]$, $\forall j = 1,2, \ldots, r$			
Time / Controlled variable	$\forall\, t_o, t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t_1 \varepsilon T$	$\forall\, t_o, t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon T x T$
$\exists\, x_{io}\ \varepsilon\ \Sigma_i$ $i = 1,2, \ldots, n$	SC(III, 1) Strongly controllable i-th state component at t_o	SC(III, 11) Strongly controllable i-th state component	SC(III, 111) Strongly controllable i-th state component for all $[t_o, t_1] \subset T$	SC(III, 1111) Normal i-th state component
$\exists\, \underset{\sim}{\tilde{x}}_o\ \varepsilon\ \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$	SC(III, 2) Strongly controllable state in $\tilde{\Sigma}$ at t_o	SC(III, 22) Strongly controllable state in $\tilde{\Sigma}$	SC(III, 222) Strongly controllable state in $\tilde{\Sigma}$ for all $[t_o, t_1] \subset T$,	SC(III, 2222) Normal state in $\tilde{\Sigma}$
$\exists\, \underset{\sim}{x}_o\ \varepsilon\ \Sigma$	SC(III, 3) Strongly controllable state at t_o	SC(III, 33) Strongly controllable state	SC(III, 333) Strongly controllable state for all $[t_o, t_1] \subset T$	SC(III, 3333) Normal state

APPENDIX I

(Continued)

Input	$u_{j[t_o,\ t_1]}, \forall j = 1,2, \ldots, r$			
Time / Controlled variable	$\exists\, t_o,\ t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t_1 \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon TxT$
$\forall\, x_{io}\ \varepsilon\ \Sigma_i$ $i = 1,2, \ldots, n$	SC(III, 4) Strongly i-th state controllable system at t_o	SC(III, 44) Strongly completely state-controllable system	SC(III, 444) Strongly totally i-th state controllable system	SC(III, 4444) i-th state-normal system
$\forall\, \underline{\tilde{x}}_o\ \varepsilon\ \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$	SC(III, 5) Strongly state controllable system in $\tilde{\Sigma}$ at t_o.	SC(III, 55) Selectively strongly completely state controllable system in $\tilde{\Sigma}$	SC(III, 555) Strongly totally state controllable system in $\tilde{\Sigma}$	SC(III, 5555) State-normal system in $\tilde{\Sigma}$
$\forall\, \underline{x}_o\ \varepsilon\ \Sigma$	SC(III, 6) Strongly state controllable system at t_o	SC(III, 66) Strongly completely state controllable system	SC(III, 666) Strongly totally state controllable system	SC(III, 6666) State-normal system

APPENDIX 2

STRUCTURAL LISTING OF OUTPUT CONTROLLABILITY CONCEPTS

Input	$u_{j[t_o,\ t_1]}, \exists j = 1,2, \ldots, r$			
Time / Controlled Variable	$\exists\, t_o,\ t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t_1 \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon TxT$
$\exists\, y_{io}\ \varepsilon\ R$ $i = 1,2, \ldots, p$	OC(I, 1) Selectively controllable i-th output component at t_o	OC(I, 11) Selectively controllable i-th output component	OC(I, 111) Selectively controllable i-th output component for all $[t_o,\ t_1]$ c T	OC(I, 1111) Selectively uniformly controllable i-th output component
$\exists\, \tilde{\underline{y}}_o\ \varepsilon\ R^q$ R^q c R^p	OC(I, 2) Selectively controllable output in R^q at t_o, $q < p$	OC(I, 22) Selectively controllable output in R^q, $q < p$	OC(I, 222) Selectively controllable output in R^q for all $[t_o,\ t_1]$ c T, $q < p$	OC(I, 2222) Selectively uniformly controllable output in R^q, $q < p$
$\exists\, \underline{y}_o\ \varepsilon\ R^p$	OC(I, 3) Selectively controllable output at t_o	OC(I, 33) Selectively controllable output	OC(I, 333) Selectively controllable output for all $[t_o,\ t_1]$ c T	OC(I, 3333) Selectively uniformly controllable output

Input	$u_{j[t_o,t_1]}$, $\exists j = 1,2, \ldots,r$			
Time / Controlled Variable	$\exists t_o,\ t_1 \varepsilon T$	$\forall t_o \varepsilon T,\ \exists T_1 \varepsilon T$	$\forall\ t_o,\ t_1 \varepsilon T$	$\forall (t,\tau) \varepsilon TxT$
$\forall y_{io}\ \varepsilon\ R$ $i = 1,2, \ldots,p$	OC(I, 4) Selectively i-th output controllable system at t_o	OC(I, 44) Selectively completely i-th output controllable system	OC(I, 444) Selectively totally i-th output controllable system	OC(I, 4444) Selectively uniformly i-th output controllable system
$\forall \tilde{y}_o\ \varepsilon\ R$ $R^q \subset R^p$	OC(I, 5) Selectively output controllable system in R^q at t_o, $q < p$	OC(I, 55) Selectively completely output controllable system in R^q, $q < p$	OC(I, 555) Selectively totally output controllable system in R^q, $q < p$	OC(I, 5555) Selectively uniformly output controllable system in R^q, $q < p$
$\forall y_o\ \varepsilon\ R^p$	OC(I, 6) Selectively output-controllable system at t_o	OC(I, 66) Selectively completely output controllable system	OC(I, 666) Selectively totally output controllable system	OC(I, 6666) Selectively uniformly output controllable system

APPENDIX 2

(Continued)

Input / Controlled Variable \ Time	$\underline{u}[t_o, t_1]$: $\exists\, t_o, t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t_1 \varepsilon T$	$\forall\, t_o, t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon TxT$
$\exists\, y_{io} \ \varepsilon\ R$ $i = 1,2, \ldots, p$	OC(II, 1) Controllable i-th component at t_o	OC(II, 11) Controllable i-th output component	OC(II, 111) Controllable i-th output for all $[t_o, t_1] \subset T$	OC(II, 1111) Proper i-th output component
$\exists\, \tilde{\underline{y}}_o \ \varepsilon\ R^q$ $R^q \subset R^p$	OC(II, 2) Controllable output in R^q at t_o, $q < p$	OC(II, 22) Controllable output in R^q, $q < p$	OC(II, 222) Controllable output in R^q for all $[t_o, t_1] \subset T$, $q < p$	OC(II, 2222) Proper output in R^q, $q < p$
$\exists\, \underline{y}_o \ \varepsilon\ R^p$	OC(II, 3) Controllable output at t_o	OC(II, 33) Controllable output	OC(II, 333) Controllable output for all $[t_o, t_1] \subset T$	OC(II, 3333) Proper output

APPENDIX 2

(Continued)

Input	$\underline{u}[t_o, t_1]$			
Controlled Variable \ Time	$\exists\, t_o,\ t_1 \varepsilon T$	$\forall\, t_o \varepsilon T,\ \exists\, t_1 \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon T x T$
$\forall\, y_{io}\ \varepsilon\ R$ $i = 1,2, \ldots, p$	OC(II, 4) i-th output controllable system at t_o	OC(II, 44) Completely i-th output controllable system	OC(II, 444) Totally i-th output controllable system	OC(II, 4444) i-th output-proper system
$\forall\, \tilde{\underline{y}}_o\ \varepsilon\ R^q$ $R^q \subset R^p$	OC(II, 5) Output controllable system in R^q at t_o, $q < p$	OC(II, 55) Completely output controllable system in R^q, $q < p$	OC(II, 555) Totally output controllable system in R^q, $q < p$	OC(II, 5555) Output proper system in R^q, $q < p$
$\forall\, \underline{y}_o\ \varepsilon\ R^p$	OC(II, 6) Output controllable system at t_o	OC(II, 66) Completely output controllable system	OC(II, 666) Totally output controllable system	OC(II, 6666) Output proper system

APPENDIX 2

(Continued)

Input	$u_{j[t_o,\ t_1]}$, $\forall j = 1,2, \ldots, r$			
Time / Controlled Variable	$\exists\, t_o,\ t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t, \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon TxT$
$\exists\, y_{io} \varepsilon R$ $i = 1,2, \ldots, p$	OC(III, 1) Strongly controllable i-th output component at t_o	OC(III, 11) Strongly controllable i-th output component	OC(III, 111) Strongly controllable i-th output component for all $[t_o,\ t_1] \subset T$	OC(III, 1111) Normal i-th output component
$\exists\, \tilde{\underline{y}}_o \varepsilon R^q$ $R^q \subset R^p$	OC(III, 2) Strongly controllable output in R^q at t_o, $q < p$	OC(III, 22) Strongly controllable output in R^q, $q < p$	OC(III, 222) Strongly controllable output in R^q for all $[t_o,\ t_1] \subset T$, $q < p$	OC(III, 2222) Normal output in R^q, $1 < q < p$
$\exists\, \underline{y}_o \varepsilon R^p$	OC(III, 3) Strongly controllable output at t_o	OC(III, 33) Strongly controllable output	OC(III, 333) Strongly controllable output for all $[t_o,\ t_1] \subset T$	OC(III, 3333) Normal output

APPENDIX 2

(Continued)

Input	$u_{j[t_o,\ t_1]}$, $\forall j = 1,2, \ldots, r$			
Time / Controlled Variable	$\exists\, t_o,\ t_1 \varepsilon T$	$\forall\, t_o \varepsilon T$, $\exists\, t_1 \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon T x T$
$\forall\, y_{io} \ \varepsilon\ R$ $i = 1,2, \ldots, p$	OC(III, 4) Strongly i-th output controllable system at t_o	OC(III, 44) Strongly completely i-th output controllable system	OC(III, 444) Strongly totally i-th output controllable system	OC(III, 4444) i-th output normal system
$\forall\, \tilde{\underline{y}}_o \ \varepsilon\ R^q$ $R^q \subset R^p$	OC(III, 5) Strongly output controllable system in R^q at t_o, $q < p$	OC(III, 55) Strongly completely output controllable system in R^q, $q < p$	OC(III, 555) Strongly totally output controllable system in R^q, $q < p$	OC(III, 5555) Selectively output normal system in R^q, $1 < q < p$
$\forall\, \underline{y}_o \ \varepsilon\ R^p$	OC(III, 6) Strongly output controllable system at t_o	OC(III, 66) Strongly completely output controllable system	OC(III, 666) Strongly totally output controllable system	OC(III, 6666) Output-normal system

APPENDIX 3

STRUCTURAL LISTING OF OBSERVABILITY CONCEPTS

Observed Variable	$y_{j[t_o,\ t_1]}$, $\exists j$, $j = 1,2, \ldots, p$			
Time / Identified Variable	$\exists\, t_o,\ t_1 \varepsilon\ T$	$\forall\, t_o\ \varepsilon T$, $\exists\, t_i \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\, (t_1, \tau) \varepsilon T x T$
$\exists\, x_{io}\ \varepsilon\ \Sigma_i$ $i = 1,2, \ldots, n$	OB(I, 1) Selectively observable i-th state component at t_o	OB(I, 11) Selectively observable i-th state component	OB(I, 111) Selectively observable i-th state component for all $[t_o,\ t_1]$ c T	OB(I, 1111) Selectively uniformly observable i-th state component
$\exists\, \underline{\tilde{x}}_o\ \varepsilon\ \tilde{\Sigma}$ $\tilde{\Sigma}$ c Σ	OB(I, 2) Selectively observable state in $\tilde{\Sigma}$ at t_o	OB(I, 22) Selectively observable state in $\tilde{\Sigma}$	OB(I, 222) Selectively observable system in $\tilde{\Sigma}$ for all $[t_o,\ t_1]$ c T	OB(I, 2222) Selectively uniformly observable state in $\tilde{\Sigma}$
$\exists\, \underline{x}_o\ \varepsilon\ \Sigma$	OB(I, 3) Selectively observable state at t_o	OB(I, 33) Selectively observable state	OB(I, 333) Selectively observable state for all $[t_o,\ t_1]$ c T	OB(I, 3333) Selectively uniformly observable state

APPENDIX 3

(Continued)

Observed Variable	$y_{j[t_o,t_1]}, \exists j,\ j = 1,2, \ldots,p$			
Time / Identified Variable	$\exists t_o,\ t_1 \varepsilon T$	$\forall t_o \varepsilon T,\ \exists t_1 \varepsilon T$	$\forall t_o,\ t_1 \varepsilon T$	$\forall (t_1,\tau) \varepsilon T x T$
$\forall x_{io}\ \varepsilon\ \Sigma_i$ $i = 1,2, \ldots,n$	OB(I, 4) Selectively i-th state observable system at t_o	OB(I, 44) Selectively completely i-th system	OB(I, 444) Selectively totally i-th state observable system	OB(I, 4444) Selectively uniformly i-th state observable system
$\forall \underline{x}_o\ \varepsilon\ \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$	OB(I, 5) Selectively observable system in $\tilde{\Sigma}$ at t_o	OB(I, 55) Selectively completely observable system in $\tilde{\Sigma}$	OB(I, 555) Selectively totally observable system in $\tilde{\Sigma}$	OB(I, 5555) Selectively uniformly observable system in $\tilde{\Sigma}$
$\forall \underline{x}_o\ \varepsilon\ \Sigma$	OB(I, 6) Selectively observable system at t_o	OB(I, 66) Selectively completely observable system	OB(I, 666) Selectively totally observable system	OB(I, 6666) Selectively uniformly observable system

APPENDIX 3

(Continued)

Observed Variable / Time / Identified Variable	$\underline{y}_{[t_o, t_1]}$			
	$\exists\, t_o, t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t_1 \varepsilon T$	$\forall\, t_o, t_1 \varepsilon T$	$\forall\, (t,\tau)$ TxT
$\exists\, x_{io} \varepsilon \Sigma_i$ $i = 1,2, \ldots,n$	OB(II, 1) Observable i-th state component at t_o	OB(II, 11) Observable i-th state component	OB(II, 111) Observable i-th state component for all $[t_o, t_1] \subset T$	OB(II, 1111) Observable i-th proper state
$\exists\, \tilde{\underline{x}}_o \varepsilon \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$	OB(II, 2) Observable state in $\tilde{\Sigma}$ at t_o	OB(II, 22) Observable state in $\tilde{\Sigma}$	OB(II, 222) Observable state in $\tilde{\Sigma}$ for all $[t_o, t_1] \subset T$	OB(II, 2222) Observable proper state in $\tilde{\Sigma}$
$\exists\, \underline{x}_o \varepsilon \Sigma$	OB(II, 3) Observable state at t_o	OB(II, 33) Observable state	OB(II, 333) Observable state for all $[t_o, t_1] \subset T$	OB(II, 3333) Observable proper state

APPENDIX 3

(Continued)

Observed Variable	$\underline{y}_{[t_o, t_1]}$			
Time / Identified Variable	$\exists\, t_o, t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t_1 \varepsilon T$	$\forall\, t_o, t_1 \varepsilon T$	$\forall\, (t,\tau)$ TxT
$\forall\, x_{io} \;\varepsilon\; \Sigma_i$ $i = 1,2, \ldots, n$	OB(II, 4) i-th state component observable system at t_o	OB(II, 44) Completely i-th state observable system	OB(II, 444) Totally i-th state observable system	OB(II, 4444) i-th state observable proper system
$\forall\, \underline{x}_o \;\varepsilon\; \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$	OB(II, 5) Observable system in $\tilde{\Sigma}$ at t_o	OB(II, 55) Completely observable system in $\tilde{\Sigma}$	OB(II, 555) Totally observable system in $\tilde{\Sigma}$	OB(II, 5555) Observable proper system in $\tilde{\Sigma}$
$\forall\, \underline{x}_o \;\varepsilon\; \Sigma$	OB(II, 6) Observable system at t_o	OB(II, 66) Observable system	OB(II, 666) Totally observable system	OB(II, 6666) Observable proper system

APPENDIX 3

(Continued)

Observed Variable	$y_{j[t_o,\ t_1]}$ 1 $j = 1,2,\ \ldots,p$			
Time / Identified Variable	$\exists\ t_o,\ t_1 \varepsilon T$	$\forall\ t_o \varepsilon T, \exists\ t_1 \varepsilon T$	$\forall\ t_o,\ t_1 \varepsilon T$	$\forall\ (t,\tau)\varepsilon T x T$
$\exists\ x_{io}\ \varepsilon\ \Sigma_i$ $i = 1,\ 2,\ \ldots,n$	OB(III, 1) Strongly observable i-th state component at t_o	OB(III, 11) Strongly observable i-th state component	OB(III, 111) Strongly observable i-th state component for all $[t_o,\tau,]$ c T	OB(III, 1111) Observable i-th normal state
$\exists\ \underline{\tilde{x}}_o\ \varepsilon\ \tilde{\Sigma}$ $\tilde{\Sigma}$ c Σ	OB(III, 2) Strongly observable state in $\tilde{\Sigma}$ at t_o	OB(III, 22) Strongly observable state in $\tilde{\Sigma}$	OB(III, 222) Strongly observable state in $\tilde{\Sigma}$ for all $[t_o,\ t_1]$ c T	OB(III, 2222) Observable normal state in $\tilde{\Sigma}$
$\exists\ \underline{x}_o\ \varepsilon\ \Sigma$	OB(III, 3) Strongly observable state at t_o	OB(III, 33) Strongly observable state	OB(III, 333) Strongly observable state for all $[t_o,\ t_1]$ c T	OB(III, 3333) Observable normal state

APPENDIX 3
(Continued)

Observed Variable	$y_{j[t_o,\, t_1]} \mid j = 1,2, \ldots, p$			
Time / Identified Variable	$\forall t_o,\, t_1 \varepsilon T$	$\forall t_o \varepsilon T,\, \exists t_1 \varepsilon T$	$\forall t_o,\, t_1 \varepsilon T$	$\forall (t,\tau) \varepsilon T x T$
$\forall x_{io} \ \varepsilon\ \Sigma_i$ $i = 1,2, \ldots, n$	OB(III, 4) Strongly i-th state observable system at t_o	OB(III, 44) Strongly completely i-th state observable system	OB(III, 444) Strongly totally i-th state observable system	OB(III, 4444) i-th state observable normal system
$\forall \underline{\tilde{x}}_o \ \varepsilon\ \tilde{\Sigma}$ $\tilde{\Sigma} \subset \Sigma$	OB(III, 5) Strongly observable system in $\tilde{\Sigma}$ at t_o	OB(III, 55) Strongly completely observable system in $\tilde{\Sigma}$	OB(III, 555) Strongly totally observable system in $\tilde{\Sigma}$	OB(III, 5555) Observable normal system in $\tilde{\Sigma}$
$\forall \underline{x}_o \ \varepsilon\ \Sigma$	OB(III, 6) Strongly observable system at t_o	OB(III, 66) Strongly completely observable system	OB(III, 666) Strongly totally observable system	OB(III, 6666) Observable normal system

APPENDIX 4

STRUCTURAL LISTING OF STATE INVARIANCE CONCEPTS

Input	$u_{j[t_o,\ t_2]}$, $\exists\, j = 1,2,\ \ldots,r$			
Time / Invariant Quantity	$\exists\, t_o,\ t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t_1 \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\,(t,\tau)\varepsilon TxT$
$x_i(t)\ \varepsilon\ \Sigma_i$, $i = 1,2,\ \ldots,n$ $\exists\, x_{io}\ \varepsilon\ \Sigma_i$	SI(I, 1) Selectively invariant i-th state component at t_o	SI(I, 11) Selectively invariant i-th state component	SI(I, 111) Selectively invariant i-th state component for all $[t_o,\ t_1] \subset T$	SI(I, 1111) Selectively uniformly in- variant i-th state component
$\underline{\tilde{x}}(t)\ \varepsilon\ \tilde{\Sigma},\ \tilde{\Sigma} \subset \tilde{\Sigma}$ $\exists\, \underline{\tilde{x}}_o\ \varepsilon\ \tilde{\Sigma}$	SI(I, 2) Selectively invariant state in $\tilde{\Sigma}$ at t_o	SI(I, 22) Selectively invariant state in $\tilde{\Sigma}$	SI(I, 222) Selectively state invariant system in $\tilde{\Sigma}$ for all $[t_o,\ t_1] \subset T$	SI(I, 2222) Selectively uniformly in- variant state in $\tilde{\Sigma}$
$\underline{x}(t)\ \varepsilon\ \Sigma$ $\exists\, \underline{x}_o\ \varepsilon\ \Sigma$	SI(I, 3) Selectively invariant state at t_o	SI(I, 33) Selectively invariant state	SI(I, 333) Selectively invariant state for all $[t_o,\ t_1] \subset T$	SI(I, 3333) Selectively uniformly in- variant state

APPENDIX 4

(Continued)

Input	$u_{j[t_o,\ t_2]}$, $\exists j = 1,2,\ \ldots,r$			
Invariant Quantity \ Time	$\exists t_o,\ t_1 \varepsilon T$	$\forall t_o \varepsilon T,\ \exists t_1 \varepsilon T$	$\forall t_o,\ t_1 \varepsilon T$	$\forall (t,\tau) \varepsilon T x T$
$x_i(t)\ \varepsilon\ \Sigma_i$, $i = 1,2,\ \ldots,n$ $\forall x_{io}\ \varepsilon\ \Sigma_i$	SI(I, 4) Selectively i-th state-invariant system at t_o	SI(I, 44) Selectively completely i-th state-invariant system	SI(I, 444) Selectively totally i-th state invariant system	SI(I, 4444) Selectively uniformly i-th state invariant system
$\underline{\tilde{x}}(t)\ \varepsilon\ \tilde{\Sigma},\ \tilde{\Sigma} \subset \Sigma$ $\forall \underline{\tilde{x}}_o\ \varepsilon\ \tilde{\Sigma}$	SI(I, 5) Selectively state-invariant system in $\tilde{\Sigma}$ at t_o	SI(I, 55) Selectively completely state-invariant system in $\tilde{\Sigma}$	SI(I, 555) Selectively totally state-invariant system in $\tilde{\Sigma}$	SI(I, 5555) Selectively uniformly state-invariant system in $\tilde{\Sigma}$
$\underline{x}(t)\ \varepsilon\ \Sigma$ $\forall \underline{x}_o\ \varepsilon\ \Sigma$	SI(I, 6) Selectively state-invariant system at t_o	SI(I, 66) Selectively completely state-invariant system	SI(I, 666) Selectively totally invariant system	SI(I, 6666) Selectively uniformly state-invariant system

APPENDIX 4

(Continued)

Input / Time / Invariant Quantity	$\underline{u}[t_o, t_1]$			
	$\exists\, t_o, t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t_1 \varepsilon T$	$\forall\, t_o, t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon T x T$
$x_i(t) \;\varepsilon\; \Sigma_i$, $i = 1,2, \ldots, n$ $\exists\, x_{io} \;\varepsilon\; \Sigma_i$	SI(II, 1) Invariant i-th state component at t_o	SI(II, 11) Invariant i-th state component	SI(II, 111) Invariant i-th state component for all $[t_o, t_1] \subset T$	SI(II, 1111) Invariant i-th proper state
$\underline{\tilde{x}}(t) \;\varepsilon\; \tilde{\Sigma}$, $\tilde{\Sigma} \subset \Sigma$ $\exists\, \underline{\tilde{x}}_o \;\varepsilon\; \tilde{\Sigma}$	SI(II, 2) Invariant state in $\tilde{\Sigma}$ at t_o	SI(II, 22) Invariant state in $\tilde{\Sigma}$	SI(II, 222) Invariant state in $\tilde{\Sigma}$ for all $[t_o, t_1] \subset T$, $1 < k < n$	SI(II, 2222) Invariant proper state in $\tilde{\Sigma}$
$\underline{x}(t) \;\varepsilon\; \Sigma$ $\exists\, \underline{x}_o \;\varepsilon\; \Sigma$	SI(II, 3) Invariant state at t_o	SI(II, 33) Invariant state	SI(II, 333) Invariant state for all $[t_o, t_1] \subset T$	SI(II, 3333) Invariant proper state

APPENDIX 4

(Continued)

Input	$\underline{u}[t_o, t_1]$			
Time / Invariant Quantity	$\exists\, t_o,\ t_1 \varepsilon T$	$\forall\, t_o \varepsilon T,\ \exists\, t_1 \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon TxT$
$x_i(t)\ \varepsilon\ \Sigma_i$, $i = 1,2, \ldots, n$ $\forall x_{io}\ \varepsilon\ \Sigma_i$	SI(II, 4) i-th state invariant system at t_o	SI(II, 44) Completely i-th state invariant system	SI(II, 444) Totally i-th state-invariant system	SI(II, 4444) i-th state invariant proper system
$\tilde{\underline{x}}(t)\ \varepsilon\ \tilde{\Sigma},\ \tilde{\Sigma} \subset \Sigma$ $\forall \tilde{\underline{x}}_o\ \varepsilon\ \tilde{\Sigma}$	SI(II, 5) State invariant system in $\tilde{\Sigma}$ at t_o	SI(II, 55) Completely state-invariant system in $\tilde{\Sigma}$	SI(II, 555) Selectively totally state-invariant system in $\tilde{\Sigma}$	SI(II, 5555) State-invariant proper system in $\tilde{\Sigma}$
$\underline{x}(t)\ \varepsilon\ \Sigma$ $\forall \underline{x}_o\ \varepsilon\ \Sigma$	SI(II, 6) State-invariant system at t_o	SI(II, 66) State-invariant system	SI(II, 666) Totally state-invariant system	SI(II, 6666) State-invariant proper system

APPENDIX 4

(Continued)

Input	$u_{j[t_o,\ t_1]}$, $\forall j = 1,2, \ldots, r$			
Invariant Quantity \ Time	$\exists t_o,\ t_1 \varepsilon T$	$\forall t_o \varepsilon T,\ \exists t_1 \varepsilon T$	$\forall t_o,\ t_1 \varepsilon T$	$\forall (t,\tau) \varepsilon T x T$
$x_i(t)\ \varepsilon\ \Sigma_i$, $i = 1,2, \ldots, n$ $\exists x_{io}\ \varepsilon\ \Sigma_i$	SI(III, 1) Strongly invariant i-th state component at t_o	SI(III, 11) Strongly invariant i-th state component	SI(III, 111) Strongly invariant i-th state component for all $[t_o,\ t_1] \subset T$	SI(III, 1111) Invariant i-th normal state
$\underline{\tilde{x}}(t)\ \varepsilon\ \tilde{\Sigma},\ \tilde{\Sigma} \subset \Sigma$ $\exists \underline{\tilde{x}}_o\ \varepsilon\ \tilde{\Sigma}$	SI(III, 2) Strongly invariant state in $\tilde{\Sigma}$ at t_o	SI(III, 22) Strongly invariant state in $\tilde{\Sigma}$	SI(III, 222) Strongly invariant state in $\tilde{\Sigma}$ for all $[t_o,\ t_1] \subset T$	SI(III, 2222) Invariant normal state in $\tilde{\Sigma}$
$\underline{x}(t)\ \varepsilon\ \Sigma$ $\exists \underline{x}_o\ \varepsilon\ \Sigma$	SI(III, 3) Strongly invariant state at t_o	SI(III, 33) Strongly invariant state	SI(III, 333) Strongly invariant state for all $[t_o,\ t_1] \subset T$	SI(III, 3333) Invariant normal state

APPENDIX 4

(Continued)

Input	$u_{j[t_o, t_1]}$, $\forall j = 1,2, \ldots, r$			
Time / Invariant Quantity	$\exists t_o, t_1 \varepsilon T$	$\forall t_o \varepsilon T, \exists t_1 \varepsilon T$	$\forall t_o, t_1 \varepsilon T$	$\forall (t,\tau) \varepsilon T x T$
$x_i(t) \ \varepsilon \ \Sigma_i$, $i = 1,2, \ldots, n$ $\forall x_{io} \ \varepsilon \ \Sigma_i$	SI(III, 4) Strongly i-th state-invariable system at t_o	SI(III, 44) Strongly completely i-th state invariant system	SI(III, 444) Strongly totally i-th state invariant system	SI(III, 4444) i-th state invariant normal system
$\underline{\tilde{x}}(t) \ \varepsilon \ \tilde{\Sigma}, \tilde{\Sigma} \subset \Sigma$ $\forall \underline{\tilde{x}}_o \ \varepsilon \ \tilde{\Sigma}$	SI(III, 5) Strongly state invariant system in $\tilde{\Sigma}$ at t_o	SI(III, 55) Strongly completely state-invariant system in $\tilde{\Sigma}$	SI(III, 555) Strongly totally state-invariant system in $\tilde{\Sigma}$	SI(III, 5555) State-invariant normal system in $\tilde{\Sigma}$
$\underline{x}(t) \ \varepsilon \ \Sigma$ $\forall \underline{x}_o \ \varepsilon \ \Sigma$	SI(III, 6) Strongly state-invariant system at t_o	SI(III, 66) Strongly completely state-invariant	SI(III, 666) Strongly totally state-invariant system	SI(III, 6666) State-invariant normal system

APPENDIX 5

STRUCTURAL LISTING OF OUTPUT INVARIANCE CONCEPTS

Input	$u_{j[t_1,\ t_2]}$, $\exists j = 1,2, \ldots, r$			
Time / Invariant Quantity	$\exists t_o,\ t_1 \varepsilon T$	$\forall t_o \varepsilon T,\ \exists t_2 \varepsilon T$	$\forall t_o,\ t_1 \varepsilon T$	$\forall (t,\tau) \varepsilon T x T$
$y_i(t)\ \varepsilon\ R$, $i = 1,2, \ldots, p$ $\exists y_{oi}\ \varepsilon\ R$	OI(I, 1) Selectively invariant i-th output component at t_o	OI(I, 11) Selectively invariant i-th output component	OI(I, 111) Selectively invariant i-th output component for all $[t_o,\ t_1] \subset T$	OI(I, 1111) Selectively uniformly invariant i-th output component
$\underline{\tilde{j}}(t)\ \varepsilon\ R^q$, $R^q \subset R^p$ $\exists \underline{\tilde{y}}_o\ \varepsilon\ R^q$, $q < p$	OI(I, 2) Selectively invariant output in R^q at t_o	OI(I, 22) Selectively invariant output in R^q	OI(I, 222) Selectively output invariant system in R^q for all $[t_o,\ t_1] \subset T$	OI(I, 2222) Selectively uniformly invariant output in R^q
$\underline{y}(t)\ \varepsilon\ R^p$ $\exists \underline{y}_o\ \varepsilon\ R^p$	OI(I, 3) Selectively invariant output at t_o	OI(I, 33) Selectively invariant output	OI(I, 333) Selectively invariant output for all $[t_o,\ t_1] \subset T$	OI(I, 3333) Selectively uniformly invariant output

APPENDIX 5

(Continued)

Input	$u_{j[t_1, t_2]}$, $\exists j = 1,2, \ldots, r$			
Time / Invariant Quantity	$\exists t_o, t_1 \varepsilon T$	$\forall t_o \varepsilon T$, $\exists t_2 \varepsilon T$	$\forall t_o, t_1 \varepsilon T$	$\forall (t,\tau) \varepsilon TxT$
$y_i(t)\ \varepsilon\ R$, $i = 1,2, \ldots, p$ $\forall y_{io}\ \varepsilon\ R$	OI(I, 4) Selectively i-th output-invariant system at t_o	OI(I, 44) Selectively completely i-th output invariant system	OI(I, 444) Selectively totally i-th output invariant system	OI(I, 4444) Selectively uniformly i-th output invariant system
$\tilde{\underline{y}}_o\ \varepsilon\ R^q$, $R^q \subset R^p$ $\forall \tilde{\underline{y}}_o\ \varepsilon\ R^q$, $q < p$	OI(I, 5) Selectively output-invariant system in R^q at t_o	OI(I, 55) Selectively completely output-invariant system in R^q	OI(I, 555) Selectively totally output invariant system in R^q	OI(I, 5555) Selectively uniformly output-invariant system in R^q
$\underline{y}(t)\ \varepsilon\ R^p$ $\forall \underline{y}_o\ \varepsilon\ R^p$	OI(I, 6) Selectively output-invariant system at t_o	OI(I, 66) Selectively completely output-invariant system	OI(I, 666) Selectively totally output-invariant system	OI(I, 6666) Selectively uniformly output-invariant system

APPENDIX 5

(Continued)

Input	$\underline{u}[t_o, t_1]$			
Time / Invariant Quantity	$\exists\, t_o,\ t_1 \varepsilon T$	$\forall\, t_o \varepsilon T,\ \exists\, t_2 \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon T x T$
$y_i(t)\ \varepsilon\ R$, $i = 1,2, \ldots, p$ $\exists\, y_{io}\ \varepsilon\ R$	OI(II, 1) Invariant i-th output component at t_o	OI(II, 11) Invariant i-th output component	OI(II, 111) Invariant i-th output component for all $[t_o, t_1] \subset T$	OI(II, 1111) Invariant i-th proper output
$\tilde{\underline{y}}_o\ \varepsilon\ R^q$, $R^q \subset R^p$ $\exists\, \tilde{\underline{y}}_o\ \varepsilon\ R^q$, $q < p$	OI(II, 2) Invariant output in R^q at t_o	OI(II, 22) Invariant output in R^q	OI(II, 222) Invariant output in R^q for all $[t_o, t_1] \subset T$	OI(II, 2222) Invariant proper output in R^q
$\underline{y}(t)\ \varepsilon\ R^p$ $\exists\, \underline{y}_o\ \varepsilon\ R^p$	OI(II, 3) Invariant output at t_o	OI(II, 33) Invariant output	OI(II, 333) Invariant output for all $[t_o, t_1] \subset T$	OI(II, 3333) Invariant proper output

APPENDIX 5

(Continued)

Input / Time / Invariant Quantity	$u_{[t_o, t_1]}$			
	$\exists\, t_o, t_1 \varepsilon T$	$\forall\, t_o \varepsilon T, \exists\, t_2 \varepsilon T$	$\forall\, t_o, t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon T x T$
$y_i(t) \varepsilon R$, $i = 1,2, \ldots, p$ $\forall y_{io} \varepsilon R$	OI(II, 4) i-th output invariant system at t_o	OI(II, 44) Completely i-th output invariant system	OI(II, 444) Totally i-th output invariant system	OI(II, 4444) i-th output invariant proper system
$\underline{\tilde{y}}_o \varepsilon R^q$, $R^q \subset R^p$ $\forall \underline{\tilde{y}}_o \varepsilon R^q$, $q < p$	OI(II, 5) Output invariant in R^q system at t_o	OI(II, 55) Completely output-invariant system in R^q	OI(II, 555) Totally output invariant system in R^q	OI(II, 5555) Output invariant proper system in R^q
$\underline{y}(t) \varepsilon R^p$ $\forall \underline{y}_o \varepsilon R^p$	OI(II, 6) Output-invariant system at t_o	OI(II, 66) Output-invariant system	OI(II, 666) Totally output-invariant system	OI(II, 6666) Output-invariant proper system

APPENDIX 5

(Continued)

Input	$u_{j[t_o,\ t_1]}$, $\forall j = 1,2, \ldots, r$			
Time / Invariant Quantity	$\exists\, t_o,\ t_1 \varepsilon T$	$\forall\, t_o \varepsilon T$, $\exists\, t_2 \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon T x T$
$y_i(t)\ \varepsilon\ R$, $i = 1,2, \ldots, p$ $\exists\, y_{io}\ \varepsilon\ R$	OI(III, 1) Strongly invariant i-th output component at t_o	OI(III, 11) Strongly invariant i-th output component	OI(III, 111) Strongly invariant i-th output component for all $[t_o,\ t_1] \subset T$	OI(III, 1111) Invariant i-th normal output
$\tilde{\underline{y}}_o\ \varepsilon\ R^q$, $R^q \subset R^p$ $\exists\, \tilde{\underline{y}}_o\ \varepsilon\ R^q$, $q < p$	OI(III, 2) Strongly invariant output in R^q at t_o	OI(III, 22) Strongly invariant output in R^q	OI(III, 222) Strongly invariant output in R^q for all $[t_o,\ t_1] \subset T$	OI(III, 2222) Invariant normal output in R^q
$\underline{y}(t)\ \varepsilon\ R^p$ $\exists\, \underline{y}_o\ \varepsilon\ R^p$	OI(III, 3) Strongly invariant output at t_o	OI(III, 33) Strongly invariant output	OI(III, 333) Strongly invariant output for all $[t_o,\ t_1] \subset T$	OI(III, 3333) Invariant Normal output

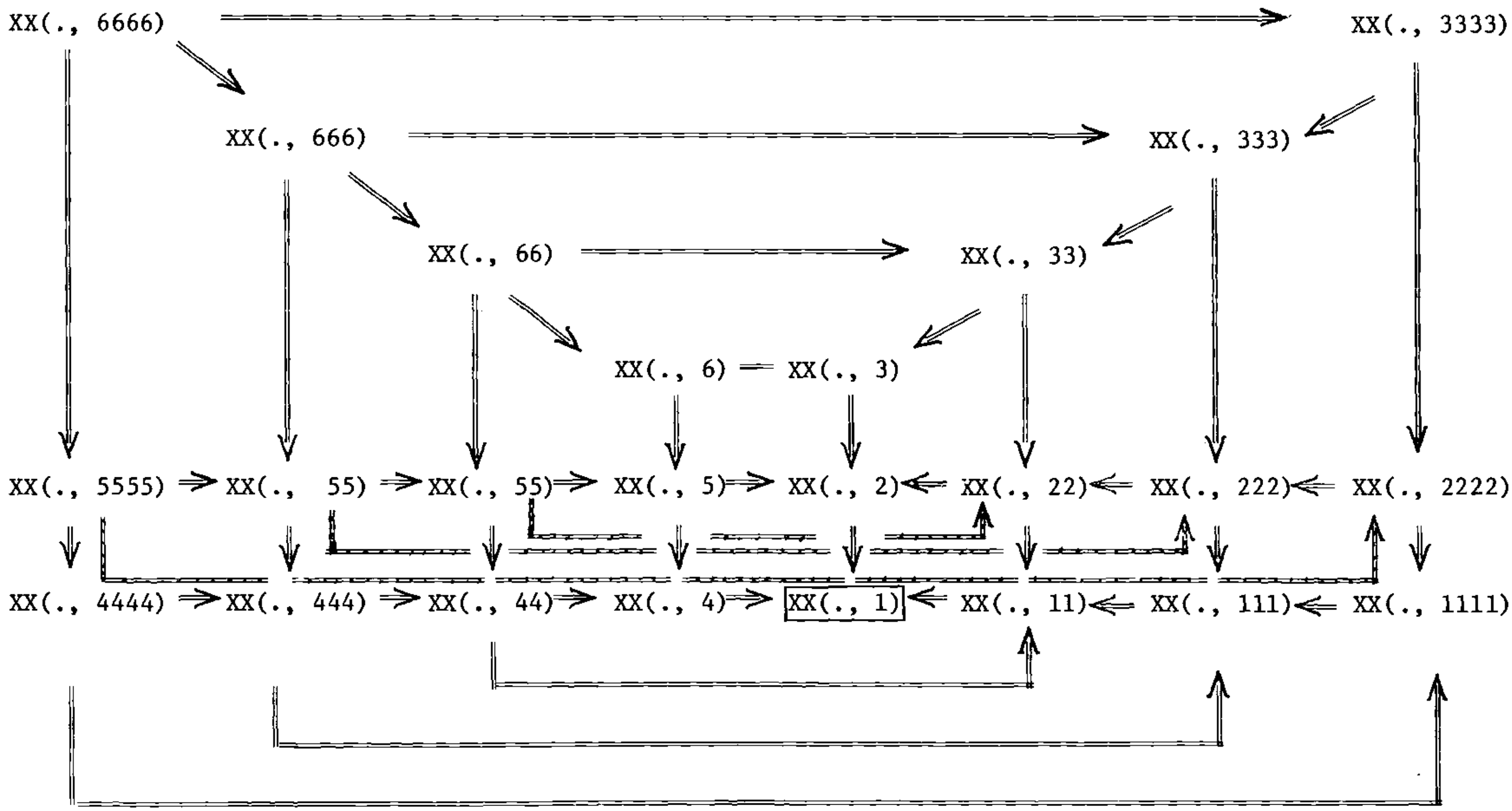
APPENDIX 5

(Continued)

Input	$u_{j[t_o,\ t_1]}$, $\forall j = 1,2, \ldots, r$			
Invariant Quantity \ Time	$\exists\, t_o,\ t_1 \varepsilon T$	$\forall\, t_o \varepsilon T$, $\exists\, t_2 \varepsilon T$	$\forall\, t_o,\ t_1 \varepsilon T$	$\forall\, (t,\tau) \varepsilon TxT$
$y_i(t)\ \varepsilon\ R$, $i = 1,2, \ldots, p$ $\forall y_{io}\ \varepsilon\ R$	OI(III, 4) Strongly i-th output invariable system at t_o	OI(III, 44) Strongly Completely i-th output invariant system	OI(III, 444) Strongly totally i-th output invariant system	OI(III, 4444) i-th output invariant normal system
$\tilde{\underline{y}}_o\ \varepsilon\ R^q \subset R^p$ $\forall \tilde{\underline{y}}_o\ \varepsilon\ R^q$, $q < p$	OI(III, 5) Strongly output invariant system in R^q at t_o	OI(III, 55) Strongly completely output invariant system in R^q	OI(III, 555) Strongly totally output invariant system in R^q	OI(III, 5555) Output invariant normal system in R^q
$\underline{y}(t)\ \varepsilon\ R^p$ $\forall \underline{y}_o\ \varepsilon\ R^p$	OI(III, 6) Strongly output invariant system at t_o	OI(III, 66) Strongly completely output invariant system	OI(III, 666) Strongly totally output system	OI(III, 6666) Output invariant normal system

APPENDIX 6

STRUCTURE OF BASIC IMPLICATIONS WITHIN THE PROPERTIES OF CONTROLLABILITY, OBSERVABILITY, AND INVARIANCE



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